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# *A refinement of the Kooi's inequality, Mittenpunkt and applications*

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# A REFINEMENT OF THE KOOI'S INEQUALITY, MITTENPUNKT AND APPLICATIONS

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(*Communicated by T. Burić*)

*Abstract.* Kooi's inequality is important in the theory of triangle inequalities, giving bound on the semiperimeter in terms of circumradius and inradius. We give one refinement, then relate the inequality to the Mittenpunk, and finally apply it to improve several known inequalities.

## 1. The inequality

Although not as widely known in the theory of triangle inequalities as the celebrated Gerretsen's inequality [7], the inequality of O. Kooi [1, 5.7] which has the same flavor - it gives bound from above on the square of the semiperimeter $s$ of a triangle in terms of its circumradius $R$ and inradius $r$ - is also significant. The inequality in question is

$$s^2 \leqslant \frac{R(4R+r)^2}{2(2R-r)}. \tag{1}$$

It is stronger than the RHS of the Gerretsen's inequality

$$16Rr-5r^2 \leqslant s^2 \leqslant 4R^2+4Rr+3r^2, \tag{2}$$

since by Euler's inequality $R \geqslant 2r$ follows that $\frac{R(4R+r)^2}{2(2R-r)} \leqslant 4R^2+4Rr+3r^2$.

## 2. Sharpened Kooi's inequality

The reference [6] where Kooi's inequality appeared for the first time is not easily accessible and apart from it we were not able to find a proof in the literature. In this section we give a sharpened version of the inequality and a proof thereof, based on the RHS of an inequality that has been labeled "fundamental inequality of a triangle" [1, 5.10]

$$\begin{aligned} 2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)} \leqslant s^2 \\ \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}. \end{aligned} \tag{3}$$

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It is sometimes called Blundon's inequality, despite its appearance in the literature being much older. The inequality is recorded in 1851 as a problem posed by C. Ramus [12], with a solution by Emile Rouché (1832-1910), who is best known for his theorem in complex analysis.

REMARK 1. Gerretsen's inequality can also be derived from (3) or by calculating distances between triangle centers and showing that $9GI^2 = s^2 - 16Rr + 5r^2$, $HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$, where $G$, $I$ and $H$ are the centroid, incenter and orthocenter of the triangle respectively. There is also a simple proof with an algebraic twist [7]. Sharp version of (3) is given in [14].

The next result is a refinement of the Kooi's inequality (1).

THEOREM 1. *For the seiperimeter $s$, circumradius $R$ and inradius $r$ of a triangle it holds*

$$s^2 \leqslant \frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{4R}.$$

*Proof.* We put $t := r/R$, $0 < t \leqslant 1/2$, $K(R,r) := \frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{4R}$ and $f(R,r) := 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}$. Then $f(R,r) \leqslant K(R,r)$ is equivalent to

$$2 + 10t - t^2 + 2(1-2t)\sqrt{1-2t} \leqslant \frac{(4+t)^2}{4-2t} - \frac{t^2(1-2t)}{4},$$

which can be rewritten as

$$2(1-2t)\sqrt{1-2t} \leqslant \frac{2(1-2t)(t^3-24t+16)}{4(4-2t)}.$$

Squaring and expanding of this last inequality gives

$$t^3(t^3 - 48t + 160) \geqslant 0,$$

which is obviously true for $0 < t \leqslant 1/2$. Hence by the fundamental inequality $K(R,r) \geqslant f(R,r) \geqslant s^2$ and the improvement of the Kooi's inequality (1) is proven. □

To understand thoroughly Kooi's inequality, it is desirable to have a geometric proof in which triangle centers are involved, a proof similar in spirit to the above mentioned proof of Gerretsen's inequalities. Such a proof we give in the next section.

## 3. Mittenpunkt and barycentric coordinates

For a point $P = (x : y : z)$ with normalized barycentric coordinates, its squared distance $OP^2$ from the circumcenter $O$ of the triangle $ABC$ with sides $a$, $b$, $c$ and circumradius $R$ is given by [13]

$$OP^2 = R^2 - a^2yz - b^2zx - c^2xy. \tag{4}$$

In the geometric proof of Gerretsen's inequalities all the classical triangle centers are involved. We consider now one lesser-known point, the Mittenpunkt $M$, which is still a household center for all the triangle geometers. That is the point which is the intersection of the lines passing through the centers of the excircles and the corresponding midpoints of the sides of the triangle (hence its name, given by the German mathematician Nagel in 1836). For more properties of $M$ see [5], where the point is denoted by $X_9$.

THEOREM 2. *The distance from the circumcenter $O$ to the Mittenpunkt $M$ of the triangle is*

$$OM^2 = R^2 - \frac{2Rs^2(2R-r)}{(4R+r)^2}.$$

*Proof.* The Mittenpunkt $M$ has normalized barycentric coordinates [16], $M = \left(\frac{a(s-a)}{D} : \frac{b(s-b)}{D} : \frac{c(s-c)}{D}\right)$, where $D = \sum a(s-a) = 2r(4R+r)$, by the well-known relation $a^2+b^2+c^2 = 2(s^2-4Rr-r^2)$, [11, p.52]. We use $\sum a(s-b)(s-c) = 2rs(2R-r)$, which follows from $ab+bc+ca = s^2+4Rr+r^2$ and $abc = 4Rrs$. Hence by (4), we get

$$\begin{aligned} OM^2 &= R^2 - \frac{abc}{D^2}\sum a(s-b)(s-c) \\ &= R^2 - \frac{2Rs^2(2R-r)}{(4R+r)^2}. \quad \square \end{aligned}$$

COROLLARY 1. *Kooi's inequality (1), which is equivalent to $OM^2 \geqslant 0$.*

## 4. Equivalent Forms of Kooi's inequality

In this section we will give few interesting equivalent forms of the Kooi's inequality. By the well-known identities

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$$

and

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} = \frac{(4R+r)^2}{s^2} - 2,$$

Kooi's inequality (1) rewritten as $\frac{(4R+r)^2}{s^2} - 2 \geqslant 2 - \frac{2r}{R}$, can be trigonometrised, giving the Garfunkel-Bankoff inequality [4]

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Another equivalent form of the inequality is

$$4R + r \geqslant s\sqrt{3 + \frac{R-2r}{R}}, \tag{5}$$

which will be used in the last section. Using the identity

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{4R+r}{s},$$

we obtain the sharpening

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geqslant\sqrt{3+\frac{R-2r}{R}}$$

of the well-known inequality $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geqslant\sqrt{3}$.

## 5. Improved Hadwiger-Finsler, Gordon's, Curry's and Leunberger's Inequality

As an application, we show that the celebrated Hadwiger-Finsler inequality [3]

$$4\sqrt{3}A+Q\leqslant a^2+b^2+c^2\leqslant 4\sqrt{3}A+3Q, \tag{6}$$

where $Q:=(a-b)^2+(b-c)^2+(c-a)^2$ and $A$ is the area of the triangle, can be improved using the Kooi's inequality to [8]

$$a^2+b^2+c^2\geqslant 4\sqrt{3+\frac{R-2r}{R}}A+Q. \tag{7}$$

Indeed, using the well-known identity $ab+bc+ca=s^2+4Rr+r^2$ and (5), we get

$$a^2+b^2+c^2-\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=4(ab+bc+ca)-(a+b+c)^2$$
$$=4r(4R+r)\geqslant 4rs\sqrt{3+\frac{R-2r}{R}},$$

which proves (7).

REMARK 2. For a proof of Hadwiger-Finsler inequality (6) based on Gerretsen's inequality (2) see [7]. The LHS of (6) can also be derived [9] from the seemingly weaker Weitzenbock inequality $a^2+b^2+c^2\geqslant 4\sqrt{3}A$.

REMARK 3. From (7) directly follows

$$ab+bc+ca\geqslant 4\sqrt{3+\frac{R-2r}{R}}A,$$

which is an improvement of the Gordon's inequality [1, 4.5] $ab+bc+ca\geqslant 4\sqrt{3}A$.

Kooi's inequality (5) together with Euler's inequality $R\geqslant 2r$ gives

$$9R\geqslant 2(4R+r)\geqslant 2s\sqrt{3+\frac{R-2r}{R}}.$$

Now $\frac{9R}{2s} \geqslant \sqrt{3+\frac{R-2r}{R}}$ is equivalent to

$$\frac{9abc}{a+b+c} \geqslant 4A\sqrt{3+\frac{R-2r}{R}},$$

which improves Curry's inequality [2] $\frac{9abc}{a+b+c} \geqslant 4A\sqrt{3}$.
Leuenberger's inequality [1, 5.22] is

$$\frac{\sqrt{3}}{R} \leqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leqslant \frac{\sqrt{3}}{2r}.$$

Its LHS is equivalent to Gordon's inequality, so it can be improved to

$$\frac{\sqrt{3+\frac{R-2r}{R}}}{R} \leqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

As a next application, we give a simpler proof of a result from [15] which seemingly generalizes (7) for $p=1$ but is actually equivalent to it.

THEOREM 3. *Let $p \geqslant 1$. Then*

$$a^{2p}+b^{2p}+c^{2p} \geqslant 3^{1-p}\left[(4A)^p\left(3+\frac{R-2r}{R}\right)^{p/2}+(a-b)^{2p}+(b-c)^{2p}+(c-a)^{2p}\right].$$

*Proof.* By Jensen's inequality applied to the convex function $f(x)=x^p$ we have $3^{p-1}\left(a^{2p}+b^{2p}+c^{2p}\right) \geqslant (a^2+b^2+c^2)^p$. The superadditivity of $f(x)=x^p$ and (7) yield

$$\begin{aligned}(a^2+b^2+c^2)^p &\geqslant (a-b)^{2p}+(b-c)^{2p}+(c-a)^{2p}+(a^2+b^2+c^2-Q)^p \\ &\geqslant (a-b)^{2p}+(b-c)^{2p}+(c-a)^{2p}+(4A)^p\left(3+\frac{R-2r}{R}\right)^{p/2}.\end{aligned}$$

Putting together the two inequalities finishes the proof. □

We conclude this note by asking for a geometric proof of the sharpening of Kooi's inequality.

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