- $\pi_n = k < n$ if, and only if, the kth and the last ball chosen have the same colour and the first $k - 1$ balls chosen have a different colour from them. Thus $P(\pi_n = k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$.

It follows that

$$\begin{aligned} E(\pi_n) &= \sum_{k=1}^{n-1} k\frac{1}{3}\left(\frac{2}{3}\right)^{k-1} + n\left(\frac{2}{3}\right)^{n-1} \\ &= \frac{1}{3}\left[\frac{1 - n\left(\frac{2}{3}\right)^{n-1} + (n-1)\left(\frac{2}{3}\right)^n}{\left(1 - \frac{2}{3}\right)^n}\right] + n\left(\frac{2}{3}\right)^{n-1}, \end{aligned}$$

using the summation at (*)

$$= 3 - 2\left(\frac{2}{3}\right)^{n-1} \to 3 \text{ as } n \to \infty.$$

The proposers, K. S. Bhanu and M. N. Deshpande, remarked that by similar methods we can derive

$$E\left(\pi_1^2\right) = 22 - \left(n^2 + 6n + 14\right)\left(\frac{2}{3}\right)^{n-1}$$

$$\text{and } E\left(\pi_n^2\right) = 15 - 3(2n + 5)\left(\frac{2}{3}\right)^n$$

from which for large n, we have $\text{var}(\pi_1) \approx \text{var}(\pi_n) \approx 6$.

Correct solutions were received from: S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett and the proposers K. S. Bhanu and M. N. Deshpande.

103.D (Ovidiu Furdui)

Let $b, c, d \in \mathbb{R}$ with $bc > 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n} \end{pmatrix}^n.$$

Answer: The limit is $\begin{pmatrix} \cosh\sqrt{bc} & \sqrt{\frac{b}{c}}\sinh\sqrt{bc} \\ \sqrt{\frac{c}{b}}\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}$, independent of d.

The following solution collates arguments used by several solvers.

Just as for limits of sequences of real numbers (but with respect to a matrix norm), if $A_n \to A$ then $\lim_{n\to\infty}\left(I + \frac{A_n}{n}\right)^n = \exp(A)$ where $\exp(A) = \sum_{n=0}^{\infty}\frac{1}{n!}A^n$. Writing $\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n}\end{pmatrix} = I + \frac{A_n}{n}$ where

$$A_n = \begin{pmatrix} 0 & n\sin\frac{b}{n} \\ n\sin\frac{c}{n} & n\left(\cos\frac{d}{n} - 1\right)\end{pmatrix} \to A = \begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} \text{ as } n \to \infty,$$

we see that

$$\lim_{n\to\infty}\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n}\end{pmatrix}^n = \exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}.$$

To calculate $\exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}$, solvers followed two paths.

- Direct calculation gives $\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}^{2n} = \begin{pmatrix} b^nc^n & 0 \\ 0 & b^nc^n\end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & b^{n+1}c^n \\ b^nc^{n+1} & 0\end{pmatrix}$ so that

$$\exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\begin{pmatrix} b^nc^n & 0 \\ 0 & b^nc^n\end{pmatrix} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\begin{pmatrix} 0 & b^{n+1}c^n \\ b^nc^{n+1} & 0\end{pmatrix}$$

$$= \begin{pmatrix} \cosh\sqrt{bc} & 0 \\ 0 & \cosh\sqrt{bc}\end{pmatrix} + \begin{pmatrix} 0 & \sqrt{\frac{b}{c}}\sinh\sqrt{bc} \\ \sqrt{\frac{c}{b}}\sinh\sqrt{bc} & 0\end{pmatrix},$$

  as required.

- Diagonalisation gives $\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} = P\begin{pmatrix} \sqrt{bc} & 0 \\ 0 & -\sqrt{bc}\end{pmatrix}P^{-1}$ where $P = \begin{pmatrix} \sqrt{b} & -\sqrt{b} \\ \sqrt{c} & \sqrt{c}\end{pmatrix}$ so that $\exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} = P\begin{pmatrix} e^{\sqrt{bc}} & 0 \\ 0 & e^{-\sqrt{bc}}\end{pmatrix}P^{-1}$

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which evaluates as $\begin{pmatrix} \cosh\sqrt{bc} & \sqrt{\dfrac{b}{c}}\sinh\sqrt{bc} \\ \sqrt{\dfrac{c}{b}}\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}$.

An alternative route used by some solvers, including the proposer Ovidiu Furdui, was to calculate $\begin{pmatrix} 1 & \sin\dfrac{b}{n} \\ \sin\dfrac{c}{n} & \cos\dfrac{d}{n} \end{pmatrix}^n$ by diagonalsing $\begin{pmatrix} 1 & \sin\dfrac{b}{n} \\ \sin\dfrac{c}{n} & \cos\dfrac{d}{n} \end{pmatrix}$ and then taking the limit as $n \to \infty$. Peter Johnson observed that the limit for $bc < 0$ corresponds to replacing $\sqrt{bc}$ by $i\sqrt{|bc|}$ in the analysis above.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett, A. Izmailov, P. F. Johnson, P. Kitchenside (3 solutions), M. Lukarevski, J. A. Mundie, I. D. Sfikas (2 solutions) and the proposer Ovidiu Furdui.

10.1017/mag.2019.129 N. J. L.

From the Editor

As you can see from the preamble to Student Problem Corner on the next page, this is Stan Dolan's last issue in charge. Stan has been the editor since taking over from Tim Cross in 2010, and he has maintained the high quality of problem setting and marking of his predecessor. He has also dealt with innovations – some more welcome than others – such as the publication of the problems on the MA website and the requirements of GDPR legislation. I am indebted to him for the commitment he has shown, but, of course, I hope he will continue to write and referee articles for the *Gazette*. I am also very grateful that Lewis Roberts has agreed to take over the role of editor of SPC.