an odd number of choices, there cannot be the same number with even size as odd size. Hence there will not be the same number of factorizations with $a + k$ even and odd, and so $n \in A$.

Since the product of two numbers whose prime factorizations have each prime factor congruent to 3 modulo 4 occurring with even power also has the same property, A is closed under multiplication.

Editorial comment. Several solvers noted that A is the set of all positive integers that can be expressed as a sum of two squares.

Also solved by R. Chapman (UK), K. Gatesman, E. J. Ionaşcu, P. Lalonde (Canada), O. P. Lossers (Netherlands), J. C. Smith, and the proposer.

An Inequality with Medians

12038 [2018, 370]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let ABC be an acute triangle with sides of length a, b, and c opposite angles A, B, and C, respectively, and with medians of length m_a, m_b, and m_c emanating from A, B, and C, respectively. Prove

$$\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} \geq 9\cos A\,\cos B\,\cos C.$$

Solution by Subhankar Gayen, Vivekananda Mission Mahavidyalaya, India. Let M be the midpoint of BC. Suppose that AM intersects the circumcircle of $\triangle ABC$ at D. By the power-of-the-point theorem, $m_a \cdot MD = a^2/4$, and two applications of the law of cosines yields $a^2/4 = (b^2+c^2)/2 - m_a^2$. Hence $b^2 + c^2 = 2m_a\,(m_a + MD)$. Since AD is a chord of the circumcircle, $m_a + MD \leq 2R$, where R is the circumradius of $\triangle ABC$. Hence $4Rm_a \geq b^2 + c^2$. Using this and the two other analogous inequalities yields

$$\begin{aligned}\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} &\geq \frac{b^2+c^2}{16R^2} + \frac{c^2+a^2}{16R^2} + \frac{a^2+b^2}{16R^2} \\ &= \frac{a^2+b^2+c^2}{8R^2} \\ &= \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2} \\ &= 1 + \cos A\cos B\cos C,\end{aligned}$$

where we have used the generalized law of sines in the second-to-last step and $A + B + C = \pi$ to obtain the last equality.

We complete the proof by showing that $1 \geq 8\cos A\cos B\cos C$. This follows from $\cos(x)\cos(y) < \cos^2((x+y)/2)$ when $x \neq y$, because this last inequality shows that $\cos A\cos B\cos C$ cannot take its maximum value on a triangle ABC unless $A = B = C = \pi/3$.

Note that the assumption that $\triangle ABC$ is acute is unnecessary and also that equality holds only when $\triangle ABC$ is equilateral.

Also solved by H. Bailey, M. Bataille (France), H. Chen, G. Fera, L. Giugiuc (Romania), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, J. F. Loverde, M. Lukarevski (Macedonia), P. Nüesch (Switzerland), P. Perfetti (Italy), C. R. Pranesachar (India), V. Schindler (Germany), D. Smith (Canada), J. C. Smith, A. Stadler (Switzerland), R. Stong, M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), L. Zhou, T. Zvonaru (Romania), GCHQ Problem Solving Group (UK), and the proposer.