Hence,

$$\int_0^\infty \frac{\log|1-x|}{x^{(1+1/k)}}\,dx = k\pi \cot\frac{\pi}{k}.$$

Editorial comment. Some solvers noted that the integral appears as equation 4.293(7) in I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (8th ed., D. Zwillinger and V. Moll, eds., Academic Press, Waltham, MA, 2015) and as equation 6.4(18) in A. Erdélyi et al., eds, *Tables of Integral Transforms* (Vol. 1, McGraw-Hill, New York, 1954). In the integral, $1/k$ can be replaced by any complex number α with $0 < \operatorname{Re}\alpha < 1$.

Also solved by K. Andersen (Canada), R. Bagby, M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, B. Bradie, M. A. Carlton, R. Chapman (U. K.) H. Chen, P. P. Dályay (Hungary), B. E. Davis, R. Dutta, N. Ghosh, M. L. Glasser, J. A. Grzesik, M. E Kuczma (Poland) & V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), M. A. Prasad (India), B. Randé (France), E. Schmeichel, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, M. Vowe (Switzerland), H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Proximity of Other Centers to the Circumcenter

11927 [2016, 723]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let O, G, I, and K be, respectively, the circumcenter, centroid, incenter, and symmedian point (also called Lemoine point or Grebe point) of triangle ABC. Prove $|OG| \le |OI| \le |OK|$, with equality if and only if ABC is equilateral.

Solution by Michael Goldenberg, Baltimore Polytechnic Institute, Baltimore, MD, and Mark Kaplan, Towson University, Towson, MD. Let a, b, c be the side lengths opposite A, B, C, respectively. Let Δ, r, R denote the area, inradius, and circumradius, respectively, of the triangle ABC.

For the first inequality, begin with the known results

$$|OG|^2 = R^2 - \frac{a^2+b^2+c^2}{9} \quad \text{and} \quad |OI|^2 = R^2 - 2rR.$$

We must therefore show $a^2+b^2+c^2 \ge 18rR$. Since

$$R = \frac{abc}{4\Delta} \quad \text{and} \quad r = \frac{2\Delta}{a+b+c}, \tag{$*$}$$

we must prove

$$\frac{a+b+c}{3}\cdot\frac{a^2+b^2+c^2}{3} \ge abc.$$

Applying the AM–GM inequality twice, we get

$$\frac{a+b+c}{3} \ge (abc)^{1/3} \quad \text{and} \quad \frac{a^2+b^2+c^2}{3} \ge (abc)^{2/3},$$

and the result follows. Equality in the first use of the AM–GM inequality holds if and only if $a = b = c$, which is to say that the triangle is equilateral.

For the second inequality, let ϕ be the Brocard angle of $\triangle ABC$. (See, for example, mathworld.wolfram.com/BrocardAngle.html.) We have

$$|OK| = \frac{R\sqrt{1-4\sin^2\phi}}{\cos\phi},$$

so we must show

$$R^2 - 2rR \leq \frac{R^2(1 - 4\sin^2\phi)}{\cos^2\phi}.$$

This is equivalent to

$$\left(1 - \frac{2r}{R}\right)\cos^2\phi \leq 1 - 4\sin^2\phi = -3 + 4\cos^2\phi,$$

which simplifies to $\cot^2\phi \geq (3R)/(2r)$. Since $\cot\phi = (a^2+b^2+c^2)/(4\Delta)$, by ($*$) this inequality is in turn equivalent to $(a^2+b^2+c^2)^2 \geq 3abc(a+b+c)$. Finally, this follows from the root-mean-square–arithmetic-mean inequality

$$\frac{a^2+b^2+c^2}{3} \geq \left(\frac{a+b+c}{3}\right)^2$$

and the ordinary AM–GM inequality. Equality holds if and only if $a = b = c$, as before.

Also solved by M. Bataille (France), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (France), R. Chapman (U. K.), I. Dimitric, S. Hitotumatu (Japan), B. Karaivanov (U. S. A.) & T. Vassilev (Canada), O. Kouba (Syria), M. Lukarevski (Macedonia), J. Minkus, R. Nandan, P. Nüesch (Switzerland), A. Stadler (Switzerland), N. Stanciu & T. Zvonaru (Romania), R. Stong, T. Wiandt, Z. Vörös (Hungary), M. Vowe (Switzerland), J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

An Arctangent Series

11932 [2016, 831]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let r be an integer. Prove

$$\sum_{n=-\infty}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) = \pi r.$$

Solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, Logroño, Spain. It suffices to prove the result for $r > 0$, because the functions sinh and arctan are odd, and the case $r = 0$ is easy. Using the identity

$$\arctan\left(\frac{x-y}{1+xy}\right) = \arctan x - \arctan y,$$

we obtain

$$\arctan\left(\frac{\sinh\ r}{\cosh\ n}\right) = \arctan\ \frac{e^{-(n-r)} - e^{-(n+r)}}{1 + e^{-2n}} = \arctan\ e^{-(n-r)} - \arctan\ e^{-(n+r)}.$$

Let $b_n = \arctan(e^{-(n+r)})$, and let S be the sum to be evaluated. We have

$$\begin{aligned} S - \arctan(\sinh\ r) &= 2\sum_{n=1}^{\infty} \arctan\left(\frac{\sinh\ r}{\cosh\ n}\right) = 2\lim_{N\to\infty}\sum_{n=1}^{N}(b_{n-2r} - b_n) \\ &= 2\sum_{n=1}^{2r} b_{n-2r} - 2\lim_{N\to\infty}\sum_{n=N+1}^{N+2r} b_{n-2r} = 2\sum_{n=1}^{2r} b_{n-2r}. \end{aligned}$$

Using the identity $\arctan z + \arctan\ z^{-1} = \pi/2$ for $z > 0$, we deduce