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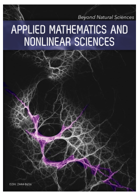

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Shapley-Folkman-Lyapunov theorem and Asymmetric First price auctions

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Abstract

In this paper non-convexity in economics has been revisited. Shapley-Folkman-Lyapunov theorem has been tested with the asymmetric auctions where bidders follow log-concave probability distributions (non-convex preferences). Ten standard statistical distributions have been used to describe the bidders' behavior. In principle what is been tested is that equilibrium price can be achieved where the sum of large number non-convex sets is convex (approximately), so that optimization is possible. Convexity is thus very important in economics.

Keywords: Shapley-Folkman theorem, asymmetric auctions, Backward shooting method, auction solving, fixed point iterations.
AMS 2010 codes: C65, D44.

1 Introduction

The Shapley-Folkman theorem places an upper bound on the size of the non-convexities (openings or holes) in sum of non-convex sets in Euclidean, $n-$ dimensional space as a function of the size on non-convexities in the sets summed and the dimension of the space, as explanation given in(Starr 2016). The bound is based on the size of non-convexities in the sets summed and the dimension of the space. When the number of sets in the sum is large, then the bound is independent of the number of sets summed and is depending on n i.e. dimension of the space. Than the size of non-convexity in the sum becomes relatively small as a proportion of the sets summed, the non-convexity becomes zero as the number of demands becomes large as the summands become large. This theorem is used to demonstrate the following properties: First, the existence of competitive equilibrium in large finite economies (Second fundamental theorem also applies with finite agents or finite time periods) with non-convex preferences. This is done by increasing marginal rate of substitution, that is by increasing the size of the proportion of the prices of at least two goods, or one indivisible good, discrete goods that can be traded only as a whole with at least two bidders. And second, this theorem is been used to demonstrate convergence

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to the set of competitive equilibria (Arrow and Hahn 1971). Further, the Shapley-Folkman lemma provides that the sum of many sets is close to being convex. In this regard, the Shapley- Folkman theorem states a bound on the distance between the Minkowski sum and its convex hull; this distance is zero if and only if the sum is convex. A set of points is said to span a point, if a point can be expressed as a convex combination (weighted average) of elements of the set of points. In case of a set of concavity of preferences, then some prices support budget line that supports two optimal baskets of goods, and then the demand for goods is disconnected. So as $n-$ dimensions increases to infinity, $1/n$ times sum of sets goes to zero. In optimization this produces an upper bound on the duality gap of separable non-convex optimization problems that involve finite sums (Aubin and Ekeland 1976).

To prove this theorem, we apply practical approach by using Asymmetric auctions, where players are randomly drawn to have different distributions and values. In this case bidder have log-concave distribution functions. The results prove that under some circumstances equilibrium price is achieved, though that equilibrium may be inefficient (high price) and non-existent (if convergence is not achieved). There exists literature in the subject of asymmetric auctions (Maskin and Riley 2000, Fibich and Gavious 2003, Guth, Ivanova-Stenzel et al. 2005, Gayle and Richard 2008, Hubbard and Paarsch 2009, Fibich and Gavish 2011, Hubbard, Kirkegaard et al. 2013).

2 The Shapley-Folkman theorem

Shapley- Folkman theorem (Starr 1969), in economics is used to extend Minkowski sums of convex sets to sums of general sets, which need not be convex (Skiena 2009):

$$\sum(A \bigoplus B) = \{a+b : a \in A, b \in B\} \tag{1}$$

Where A and B are sets of location vectors or radius vectors. Shapley- Folkman theorem may be represented this way:

Theorem 1. *Lets suppose that $x \in con(A_1 + \cdots + A_I)$, where$A \in R^L$, So we can write $x = a_1 + \cdots + a_i$, where $a_i \in conA_i, \forall i, a_i \in A_i, \forall L, \subsetneq, \forall i.$*[a]

Previous theorem originates from Carathéodory's Fundamental Theorem (Eckhoff 1993): Each point in the convex hull of a set S in R^n is in the convex combination of $n+1$ or fewer points of S. Convex hull[b] the is given as:

$$con \equiv \{\sum_{j=1}^{N} \lambda_j p_j : \lambda_j \geq 0 \forall j, \sum_{j=1}^{N} \lambda_j = 1\} \tag{2}$$

Suppose that $x \in conA$, where $A \in R^L$, $\exists a_1 \ldots, a_{n+1} \in A, \in con(a_1, \ldots, a_{n+1})$. More convenient approach to the statement of this theory is presented by Khan and Rath (Khan and Rath 2013):

Let $\mathbb{R}^L$ be an L dimensional Euclidean space, then let A_i denote the its convex hull for any$A \subseteq R^n$, now $\emptyset \neq A_i \subseteq R^n$ and $x \in con\sum_{i=1}^{n} A_i$, then $\sum_{i=1}^{n} x_i = x$, and $x \in conA_i, \forall i, x_i \in A_i$, for at least $n-m$ indices of i.

Let $\Psi = \emptyset$[c], and then $a, b \in \Psi \rightarrow a \cup b \in \Psi, a, b \in \Psi \Rightarrow a \backslash b \in \Psi$. [d] This contains logical values (Boolean Algebra), and one can define finitely additive measure $\mu : \Psi \rightarrow R^L$, i.e.$\mu(a \cup b) = \mu(a) + \mu(b)$,whenever $a, b \in \Psi, a \cap b = \emptyset$. Partially ordered subset of partially ordered set i.e. supremum of μ is given as : $sup\{|\mu_i(a)| : a \in \Psi\} < \infty, 1 \leq i \leq m$[e], if $m = 1$ it is called finitely additive scalar measure. And we can define:$|\mu| = \mu^+ + \mu^-$. A field Ψ is an $F-$ algebra, i.e.$F : \Psi \rightarrow \Psi$ is an endofunctor of category Ψ, then an $F-$algebra is a tuple

[a] $conA_i$ for a theory A_i means A_i is consistent

[b] A convex hull is the smallest polygon that encloses a group of objects, such as points.

[c] Ψ is a field of subsets.

[d] The relative complement of a with respect to a set b

[e] This is the least element in $\Psi \geq \forall a_i, b_i$

(A,a), where A is an object of Ψ and a is isomorphism $F(A) \to A$. Ψ also is a $F-$ algebra if for any increasing sequence $\{A_n\}$ and any decreasing sequence $\{B_n\}$, and $A_n, B_n \in \Psi, A_n \subseteq B_n, \forall n, \exists C \in \Psi, A_n \subseteq C \subseteq B_n \forall n$ (Seever 1968, Armstrong and Prikry 1981). Ψ also, is a σ -algebra if $\cup_{n=1}^{\infty} A_n \in \Psi, A_n \in \Psi$, in such a case $\Psi, (T, \Psi)$ is a measurable space. Every$F-$algebra is σ-algebra: if Ψ is a σ-algebra and if there is increasing sequence $\{A_n\}$ and decreasing sequence $\{B_n\}$ a decreasing sequence both in Ψ and $A_n \subseteq B_n, \forall n$, then $\cup_{n=1}^{\infty} A_n$ or $\cup_{n=1}^{\infty} B_n$ can be in role of C. Now if so,$\mu : \Psi \to R^L$ is an countably additive measure if $\mu(\emptyset) = 0$, and $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{i=1}^{n} \mu(A_n)$, when $\{A_n\}$ is a sequence of disjoint pairs of sets in Ψ. The measure μ purely atomic if there is scalar measure λ such that $\lambda \ll \mu$, and if $\lambda(A) = 0$, for every measurable set for which $|\mu|(A) = 0$. And if there is a sequence $\{E_k\}$ such that $T = \cup_{k=1}^{\infty} E_k, \forall E_k \in \forall \mu_i, i = 1, \dots, m$. This is called an atomic or only positive measure (Bogachev 2007, Aliprantis and Border 2013, Halmos 2013, Hewitt and Stromberg 2013). Proof of the theorem was most simple in the one provided in (Zhou 1993):

Proof. Let $x \in Co(A)$ has a representation $x = \sum_{i=1}^{n} y_i$ and $y_i \in Co(A_i), \forall i$, and let $y_i = \sum_{i=1}^{n} a_{ij} y_{ij}, a_{ij} > 0, \sum_{i=1}^{n} a_{ij} = 1, y_{ij} \in A_i$. Constructed z vectors are given as: $z = \sum_{i=1}^{n} \sum_{j=1}^{l} b_{ij} y_{ij}, b_{ij} \geq 0$, and $(m+n) b_{ij} \geq 0$.From previous expression $x = \sum_{i=1}^{n} \sum_{j=1}^{l} b_{ij} y_{ij}, b_{ij} = 1, \forall i$. Now, $x = \sum_{j=1}^{l} b_{ij} y_{ij}, x = \sum_{i=1}^{n} x_i, x_i \in Co(A_i), \forall i$.Because there are $(m+n) b_{ij} \geq 0$ in total, there is at least one $b_{ij} \geq 0, \forall i$, and there are at most m indices i that have more than one $b_{ij} > 0$, and $x_i \in A_i$ for at least $(n-m)$ indices i. Preposition also here is continuous function (continuity condition) i.e. the condition for continuity, where states that f is said to be continuous on R^l if (Robbin, Rogers et al. 1987):

$$\forall x_0 \in R^l \forall \varepsilon > 0 \exists \delta > 0 \forall x \in R^l [|x - x_0| < \delta| \Rightarrow f(x) - f(x_0) < \varepsilon|] \tag{3}$$

Also, if there are n commodities, and a nonnegative orthant Θ of Euclidean spaceE^n is introduced, then the sets $\{x : x \succeq_c y\}$ and $\{x : y \succeq_c x\}$ are closed. Here $\succeq_c$ are preferences of a trader in a pure exchange economy (Starr 1969). The assumption of convexity assumes that if $\succeq_c y$, then $\lambda x + (1-)x \succeq_c y$, this means that any weighted average or convex combination of x and y is preferred to $y, 0 \leq \lambda \leq 1$. Each trader has initial endowment bundle and stars with a positive amount of some good $x_c > 0$.

Now, will introduce spanability assumption. Let, $\vartheta \in \Theta$, and $x \in A_c(\vartheta)$,then there is a set of no more than $n+1$ points x_c of $A_c(\vartheta)$ and $x = \sum \lambda_c x_c$, where $\lambda_c \geq 0$, for all c, and $\sum \lambda_c = 1$. Spanability of functions is an important concept in mathematical economics, further explained in Hüsseinov (Hüsseinov 1997):

Theorem 2. *Spanability theorem: Let $f : R^n \to \bar{R}$[f] is a lower semicontinuous bounded from below function such that epigraph of a function given as,*
$epi(f) = \{(x, \vartheta) | x \in X, \vartheta \in R, f(x) \leq \vartheta\} \subset R^{n+1}$, [g] does not contain lines. We were that the function is convex if its epigraph $epi(f)$ is convex set. And, furthermore $x = \sum_{i=1}^{m} \lambda_c x_c, \exists x_i, \dots, x_m \in R^n$, and

$$\forall (\lambda_1, \dots, \lambda_m) > 0, \lambda_1 + \cdots + \lambda_m = 1,\ f*(x) = \sum_{i=1}^{m} \lambda_c f(x_c).$$

[h]*Function would be spannable also in this corollary:* $\lim_{||x|| \to \infty} \frac{f(x)}{||x||} = \infty$, *then f is spannable.*

3 Lyapunov theorem

The Shapley-Folkman theorem is a discrete counterpart to the Lyapunov theorem on non-atomic measure (Starr 2016), Lyapunov theorem can be best presented as in by Grodal (Derigs 2009). Let is consider an economy of finite non-negative atomless measures i.e.$\mu = (\mu_1, \dots, \mu_n)$ on the measurable space $(A, \mathbb{A})$.In the previous expression $\mathbb{A}$ is a set of finite vector defined measures on V,which represents a set of coalitions of consumers as in

[f] Here $\bar{R} = R \cup \{-\infty, +\infty\}$ affinely extended real numbers
[g] Epigraph or supergraph of a function $f : R^n \to R$ is
[h] $f*$ is the greatest lower semicontinuous convex.

the works of Vind (Vind 1964). Here also V is a σ- field of subsets C. By, $\bar{A}_0$ is given the resource allocation (set of possible allocations) in the economy $\{a \in \mathbb{A} | a\, nonatomic\}$. Then, the range of atomless measures is: $R(\mu) = \{x \in R^n | \exists C \in \mathbb{A}\}$ where $x_h = \mu_h(C), h = 1, \ldots, n$, is a compact and convex subset of R^n. Lyapunov theorem is used to make conclusions about the trajectories of the system $\dot{x} = f(x)$. System is globally asymptotically stable if $x(t) \to x_e$, as $t \to \infty$. System is locally asymptotically stable near or at x_e, if there is an $R > 0$, subject to $||x(0) - x_e|| \leq R \Rightarrow x(t) \to x_e, t \to \infty$. In previous expressions x_e is an equilibrium point and $x_e \in R^n$. Now some function $V : R^n \to R$ is positive and definite if: $V(z) \geq 0, \forall z$. Now if there is a nonlinear system $x = f(x)$, and function $V : R^n \to R$, one can define; $\dot{V} : R^n \to R$, and one can write: $\dot{V}(z) = \nabla V(z)^T f(z)$, or $\frac{dV}{dt}(x(t))$, when $z = x(t)$, and $\dot{x} = f(x)$. Lyapunov theorem imply states that if there exist such function $V : R^n \to R$ that satisfies V and $\dot{V}$. Function $V : R^n \to R$ is called Lyapunov function. Trajectories of this function are bounded from zero to t. This can be written as:

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(t))d\tau \tag{4}$$

About Lyapunov stability following conditions should apply: $\dot{V}(z) < 0, \forall z \neq 0$, and $\dot{V}(0) = 0$, if $z = 0$. Proof of this theorem assumes that $x(t) \to 0, \dot{V}(x(t)) \to \varepsilon$, and $\varepsilon > 0$. And for ε following applies: $\varepsilon \leq V(x(t)) \leq V(x(0))$. And a subset C is closed and bounded $C = \{z | \varepsilon \leq V(x(z)) \leq V(x(0))\}$. $\dot{V}$ is also assumed to be continuous, and $sup_{z \in C} \dot{V} = -a < 0$. And, since $\dot{V}(x(t)) < -a, \forall t$, one can rewrite Equation 4 (prethodo ravenstvo) as:

$$V(x(t)) = V(x(0)) + \int_0^T \dot{V}(x(t))d\tau \leq V(x(0)) - aT \tag{5}$$

and since $T > V(x(0))/a$, and $V(x(0)) < 0$, this means that $x(t) \to 0$, and that $\dot{x}$ is globally asymptotically stable (Brouwer 1912). Now if$(T, \mho, \mu)$ is a measurable space and $0 < \mu(A) < \mu(B)$, where $A \in \mho$ and $B \in \mho, B \subset A$. In previous expressions $\mho$ is a σ-algebra, and the nonatomicity of μ is shown is equivalent to (Tardella 1990):

$$\forall a \in [0,1], \forall A \in \mho, \exists B \in \mho, B \subset A, \mu(B) = a\mu(A) \tag{6}$$

The main idea of Lyapunov theory is that $\dot{V}(z) < 0$ or $\dot{V}(x) < 0$, along the trajectories of the system, than $V(x)$ will $\downarrow, t \to \infty$ (Hokayem, Mastellone et al. 2006). If for instance one considers nonlinear system in the following form:

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} -x_1 + 3x_1^2 x_2 \\ -x_2 \end{bmatrix} \tag{7}$$

And the candidate Lyapunov function given as: $V(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$. And $\lambda_1, \lambda_2 > 0$. And $V(x) \to \infty$ as $||x|| \to \infty$. The derivative of V along the trajectories is given as:

$$\dot{V} = 2\lambda_1 x_1(-x_1 + 3x_1^2 x_2) + 2\lambda_2 x_2(-x_2) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 6\lambda_1 x_1^2 x_2 \tag{8}$$

Henceforth, if $\dot{V}(x) < 0$, $V \downarrow$ along the solution of $\dot{x} = f(x)$. Local minimum of a convex function is given as some point x^* such that: $\exists \varepsilon > 0$, and $f(x)^* < f(x), \forall x$, all the feasible x (Mishra, Wang et al. 2008).

4 The second fundamental welfare theorem and nonconvexities

Second fundamental theorem is giving conditions under which a Pareto optimal allocation can be supported as a price equilibrium with lump-sum transfers, i.e. Pareto optimal allocation as a market equilibrium can be achieved by using appropriate scheme of wealth distribution (wealth transfers) scheme (Mas-Colell, Whinston et al. 1995).Assumption of convexity (in technology, preferences), is crucial for the establishment of the second welfare theorem. Second fundamental welfare theorem can be specified as follows: Given an economy such as; $(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j, \}_{j=1}^J, \bar{\omega})$ [i], allocation is given with $(x*, y*)$, and a price vector $p \neq 0$, (here $p = p_1, \ldots, p_n$), and

[i] We assume that $\sum_i \omega \ll 0$ (level of income or wealth is much larger than zero)

this combination constitutes price quasi-equilibrium with transfers $(\omega_1,\ldots,\omega_I)$, subject to the following budget constraint: $\sum_i \omega_i = p\bar{\omega} + \sum_i py_j^*$, where $\forall j, y_j^*, max_j py_j^*, py_i \leq py_j^*, \forall y_j \in Y_j$. And $\forall i, \exists x_i \succeq_i x_i^* \rightarrow px_i \geq \omega_i$. And $\sum_i x_i^* = \omega + \sum_j y_j^*$. Under the assumption of locally non-satiated preferences ($\succeq$ on X[j] is locally nonsatiated if $\forall x \in X, \varepsilon > 0, \exists y \in X, ||y-x|| < \varepsilon$, $u(y) > u(x)$), we have following equality $px_i = \omega_i$. This theorem holds if preferences are convex i.e.: The set $A \subset R^n$ is convex compact and nonempty set if $\lambda x + (1-\lambda)x' \in A, x, x' \in A$ and $\lambda \in [0,1]$. There is a theorem that gives sufficient conditions for the existence of hyperplane separating sets, that is the Separating hyperplane theorem[k]. A hyperplane is the set:

$$H(p,\alpha) = \{x \in R^n | pH0 = \alpha\} \tag{9}$$

Hyperplane in R^n can be described by an equation $\sum_{i=1}^n p_i x_i = \alpha$, here vector $p \in R^n$ is a non-zero price vector, and α is scalar (Simon and Blume 1994, Yu and Phillips 2018). The vector p is then said to be normal to the hyperplane H. If one defines two points $(x*, y*) \in H(p,\alpha)$, and by defining, $p \cdot x* = \alpha$ and, $p \cdot y* = \alpha$, in other words vector p is orthogonal to the line segment $(x* - y*)$ i.e. $p \cdot (x* - y*) = 0$. By picking arbitrary points previous result can be generalized to all points i.e. line segments in (p,α), or that p is orthogonal to $H(p,\alpha)$. Open half-spaces in the hyperplane are defined by inequalities: $\{x \in R^n | pH0 \leq \alpha\}$ or $\{x \in R^n | pH0 \geq \alpha\}$. First expression, from previous two is example of budget set. If the preferences are assumed convex then every u_i is concave and under this hypothesis every set U_i is concave. First, a vector $x = (x_1,\ldots,x_n) \in R^n$ is an allocation if $\sum_i x_i \leq \omega$, (Mas-Colell 1989).In previous expression $\omega \in R_+^n$ is a total endowment of commodities, and production is allowed. Utility function is given as; $U = \{u \in R^n : u \leq u(x)\}$. Previous function can be generalized that if preferences are nonconvex i.e. they are concave and their function is concave i.e.$f : A \rightarrow R$ is nonconvex, in other expression this is translated to:$\{(x,y) \in R^{n+1} : y \leq f(x), x \in A\}$, and the last set is convex. Let is suppose that $x \in B$ is an extreme point of the convex set $\subset R^n$, this means that $x \neq \lambda x + (1-\lambda)x', x, x' \in A$ and $\lambda \in [0,1]$. Separating hyperplane theorem can be stated as follows: Let is suppose that $B \subset R^n$ is a convex and closed set and $x \notin B, \exists p \in R^n, p \neq 0, \alpha \in R, p \cdot x > \alpha, p \cdot y < \alpha, \forall y \in B$. Convex sets $A, B \subset R^n$ are disjoint $A \cap B = \emptyset, \exists p \in R^n, p \neq 0, p \cdot x > \alpha, p \cdot y < \alpha, \forall y \in B$.Then there is a hyperplane that separates A and B, leaving them on different sides of it. In support of this theorem if $B \subset R^n$ is convex and $x \notin intB, \exists p \in R^n, p \cdot x \geq p \cdot y, p \neq 0$. If A and B are convex, $A - B \notin 0, A \cap B = \emptyset$. Let is say that $S \in R^m$ if $z*$ is a boundary point of set $S, \exists p \neq 0, z \in S \rightarrow p \cdot z \leq p \cdot z*$, proof of this will come from a simple lemma: if S is a closed and convex set [l], $x \in S$, and if b is the boundary of this set, then there exists scalar $\alpha \neq 0$ such that: $x \in S \rightarrow \alpha x \leq \alpha b$. Previous theorem ($S, \exists p \neq 0, z \in S \rightarrow p \cdot z \leq p \cdot z*$, where p is an $m-$ dimensional price vector) holds if $m = 1$, this theorem is also true when $m = n+1$ (Fibich and Gavish 2011). Here $n+1$ is the dimension of S production set. Now z is $n+1$ dimensional vector, x is n dimensional vector and y is one dimensional scalar: $z = (y,x), n$ -dimensional set is: $X(y) \equiv \{x | (y,x) \in S\}$. Now, from these two convex sets $S, X(y)$ follows that: $(\lambda y + (1-\lambda)y', \lambda x + (1-\lambda)x') \in S$, and $x \in X(y*) \rightarrow p_x x \geq p_x x*$. Now for one dimensional space we use following lemma to proof separating hyperplane theorem: $f(y) - f(a) \geq f'(a)(y-a), \forall y \in A$. If we want to minimize some function let say cost function then we have: $c_{p,x}(y) = min_{x \in X(y)} >_x \cdot x$, or $c_{p,x}(y) = p_x x*$. And because cost function is a minimal function $c_{p,x}(\lambda y + (1-\lambda)y') \leq p_x(\lambda \hat{x} + (1-\lambda)\hat{x}') = \lambda c_{p,x}(y) - (1-\lambda)c_{p,x}(y)$. From, previous, it follows that $c_{p,x}(y) - c_{p,x}(y*) \geq p_y(y - y*)$, and since by definition $(y,x) \in S \Rightarrow c_{p,x}(y) \leq p_x \cdot x$. From previous, $(y,x) \in S \Rightarrow p_x \cdot x - p_x \cdot x* \geq p_y(y - y*)$. And, $n+1$ dimensional vector space is given by: $p \equiv p_y - p_x$, since $p_x \neq 0 \rightarrow p \neq 0$. These last expressions are the same as: $z \in S \rightarrow p \cdot z \leq p \cdot z*$. These results can also be proved with a Hahn-Bannach separated hyperplane theorem. Convex sets $A, B \subset X^n$ are disjoint $\cap B = \infty$, in a topological vector space X. And presumptions are: If A is open $\exists \Lambda \in X^*$ (linear map) and $\lambda \in R$

[j] Preference relation $\succeq$ is a relation $\succeq \subset R_+^l \times R_+^l$. With properties $x \succeq x, \forall x \in R_+^l$ (reflexivity), $x \succeq y, y \succeq z \Rightarrow x \succeq z$ (transitivity),$\succeq$ is a closed set (continuity), $\forall (x \succeq y), \exists (y \succeq x)$ (completeness), given $\succeq, \forall (x \ll 0)$ the at least good set $\{y : y \succeq x\}$ is closed relative to R^l (boundary condition), A is convex, if $\{y : y \succeq x\}$ is convex set for every $y, ay + (1-\lambda)x \succeq x$, whenever $y \succeq x$ and $0 < a < 1$, Mas-Colell, A. (1986).

[k] In geometry hyperplane of an n dimensional vector space V is a subspace of a $n-1$ dimension, or equivalently of codimension 1 in V

[l] Its complement is an open set. Closed set is defined as a set that contains all of its limit points.

such that: the real part of the complex number is given as $Re\Lambda x \leq \lambda \leq Re\Lambda y, \forall x \in A, \forall y \in B$. If A is compact [m], B is closed, and λ is locally convex, $\exists \Lambda \in X^*, \lambda_1 \in R, \lambda_2 \in R$, and $Re\Lambda x < \lambda_1 < \lambda_2 < Re\Lambda y, \forall x \in A, \forall y \in B$.In real numbers set only $Re\Lambda = \Lambda$. Real scalars are assumed $A_0 \in A, b_0 \in B$, and convex neighborhood is given as: $C = A - B + x_0$,since $A \cap B = \infty, x_0 \notin C$, and Minkowski functional is given as: $p(x_0) \geq 1$. In particular $\Lambda \leq 1$ on $C\Lambda \leq 1$ on $-C$. Now, this means that $|\Lambda| \leq 1$,on the neighborhood $C \cap (-C)$ of 0, and $\Lambda \in X^*$. Now if $\alpha \in A, b \in B$,will give : $\Lambda a - \Lambda b + 1 = \Lambda(a - b + x_0) \leq p(a - b + x_0) < 1$.Now, since $\Lambda x_0 = 1, a - b + x_0 \in C$, and C is open, $\Lambda a \neq \Lambda b, \Lambda a < \Lambda b, So \Lambda(A), \Lambda(B)$ are closed disjoint sets (Rudin 2006). There is one supporting property more, following Mas-Colell, it states that :for $\forall p \neq 0$, the subspace $T_p = \{v \in R^n : p \cdot v = 0\}$, and this subspace is called hyperplane perpendicular to p (Mas-Colell 1989). The convex set is $\{v \in R^n : p \cdot v \leq 0\}$, and it is a half space below T_p. Convex set is, where $x \in A, if A - \{x\} \subseteq x_n > 0 (\geq 0)$. And, one can write that: $\{p \cdot y : y \in A\} \leq p \cdot x$, i.e. $p \cdot A \leq p \cdot x$, if $p \cdot y \leq p \cdot x, \forall y \in A, y \neq x$, one can say that p supports strictly. Therefore, supporting hyperplane theorem:

Theorem 3. *Supporting hyperplane theorem: If $A \subset R^n$ is convex and $x \in \partial A$, the A is supported at x by some $p \neq 0$. If A is closed, and $x = 0$, then the support can be strict.*

Now, for a concave function $f : (a,b) \to R$ is continuous in $IntA$. This function $f : (a,b) \to R$ is concave in the interval (a,b), if for every $x_1, x_2 \in (a,b), a \in (0,1)$, it follows $f(ax_1 + (1-a)x_2) < af(x_1) + (1-a)f(x_2)$.If the function is continuous and twice differentiable i.e. C^2 on the open interval (a,b), then this function is concave on (a,b) if: $\forall x \in (a,b), f''(x) < 0$. Or a C^2 function: $g : A \to R^n$ on the open and convex set $A \subset R^n$ is concave if and only if $\partial^2 f(x) < 0$ and is semidefinite for all x,then f is strictly concave. In the literature of this king very important term is marginal cost pricing equilibrium which is a family of consumption, production plans, lump sum taxes and prices such that such that households are maiming their utility subject to their budget constraints and firms production plans that satisfy the FONCs (First order necessary conditions), for smooth optimization and for functions as: $f : R^l \to R^n$ and $h : R^l \to R^m$ are C^1, constraint set is given as : $E = \{x \in R^l : h(x) \geq 0\}$ (Brown 1991).
Some $x \in E$ is a local weak maximum of f subject to h. Now FONCs are:

1. If x is a local weak maximum of f subject to h, then there are$(\lambda, \mu) \in R^n_+ \times R^m_+$ so that:
   (a) $(\lambda, \mu) \neq 0$
   (b) $h_j(x) < 0, \mu_j = 0$
   (c) $\sum_{i=1}^n \lambda_i \partial f_i(x) + \sum_{j=1}^m \mu_j \partial h_j(x) = 0$ (Mas-Colell 1989)
2. SONCs (second order necessary conditions) state that if f and h are C^2, and $x \in E$ satisfies the FONCs with respect to $(\lambda, \mu) \in R^n_+ \times R^m_+$, than we will consider the bilinear form: $B = \sum_{i=1}^n \lambda_i \partial^2 f_i(x) + \sum_{j=1}^m \mu_j \partial^2 h_j(x)$, and the cone is presented by; $K = \{v \in R^l : \partial f_i(x) \geq 0, \lambda_i \partial f_i(x) \cdot v = 0, \forall i, \partial h_j(x) \cdot v \geq 0, \mu_j \partial h_j(x) \cdot v = 0, \forall j, and\, h_j(x) = 0\}$. Now if x is local maximum, $\lambda > 0$ and rank $\{\partial f_i(x), \partial h_j(x) : i \leq n, j \leq m, h_j(x) = 0\} = n + \{j : h_j(x) = 0\} - 1$, then B is negative and semidefinite. If B is negative and semidefinite on K then x is local maximum.

For the MCP equilibrium it is important to note that if all firms have convex technologies with zero vector, then MCP equilibrium becomes Walrasian equilibrium. Now from previous, if f is strictly non-convex (concave), and the feasible set A is convex, then the maximizer x^* is unique. In the proof of this let is suppose that there exist two maximizers, x and x', then we have $\lambda x + (1-\lambda)x' \in A$, by the strict definition for concavity $0 < \lambda < 1$. Now, $f(\lambda x + (1-\lambda)x') > \lambda f(x) + (1-\lambda)f(x') = f(x) = f(x')$. As an example let is take one consumer problem let say: $max_{x_1,x_2} x_1^{1/2} x_2^{1/2}$, subject to: $x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0$. So the feasible set

[m] A subset of Euclidean space in particular is called compact if it is closed and bounded. This implies, by the Bolzano鈥揥eierstrass theorem, that any infinite sequence from the set has a subsequence that converges to a point in the set.

$A = \{x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$, and this is compact set [n], and the function $x_1^{1/2} x_2^{1/2}$ is continuous, so by the Bolzano - Weierstrass theorem, that states that any infinite sequence from the set, every infinite subset of S has a limit point (this not necessary lies in the subset) (Green and Heller 1981). In general case for a given bifunction $B : H \times H \rightarrow R$, one considers finding a solution $u \in K$, where K is a closed and convex set, and: $B(u, v-u) \geq 0, \forall v \in K$, this is called bifunction variation inequality (Noor, Noor et al. 2012). In, previous expressions H is a Hilbert space. Now if there are two given bifunctions with their inner products $F(\cdot,\cdot), B(\cdot,\cdot) : H \times H \rightarrow R$. The problem of finding $u \in K$ is called nonconvex bifunction equilibrium variational inequality: $F(u,v) + B(u, v-u) + \varphi||v-u||^2 \geq 0, \forall v \in K$. Proposed iterative method, here is convergence that requires partially relaxed strongly monotonicity. A monotonic function is a function which is either entirely nonincreasing or nondecreasing. A function is monotonic if its first derivative (which need not be continuous) does not change sign. If a function $f : X \rightarrow Y$ is a set function from a collection of sets X to an ordered set Y, then f is monotone whenever $A \subseteq B$ as elements of $X, f(A) \leq f(B)$ (Royden and Fitzpatrick 2017). The proposed bifunction can be monotone Type (I) and Type II. Type (I) monotonicity of proposed bifunction is given by the : $T(\cdot,\cdot) : H \times H \rightarrow R$ this bifunctions is said to be monotone Type (I) with respect to the operator g if and only if $T(u, g(v) - g(u)) + T(v, g(u) - g(v)) \geq 0, \forall u, v \in H$.And a bifunction :$F : H \times H \rightarrow R$ is said to be monotone of type (II) with respect to the operator g if and only if : $F(g(u), g(v)) + F(g(v), g(u)) \leq 0, \forall u, v \in H$ (Noor, Noor et al. 2012). The sum of large number of non-convex sets is convex (approximately) (Starr 2011, Starr 2016). A typical non-convex set contains a hole for indentation. Having this in mind one can write Shapley-Folkman lemma:

Lemma 4. *Shapley-Folkman lemma: Let $S^1, S^2, S^3, \dots S^m \subseteq \cup_{S(x,y^i)ED^n} C$ [o], and these are nonempty compact sets. And, $x = con(S^1, S^2, S^3, \dots S^m), \forall i = 1, 2, \dots, m, \exists y^i = con(S^i), \sum_{i=1}^m y^i = x, y^i \in S^i$. With utmost n exceptions.*

From here follows Shapley -Folkman theorem in general terms:

Theorem 5. *Shapley-Folkman theorem: $rad(S) \equiv inf_{x \in R^n} >up_{y \in S}>x - y|, S \subset R^n, rad(S) \leq L, \forall S \in C, rad(S)$ is the radius of the smallest ball (set) containing S. And F is a family of compact subsets $S \subset R^n$, and $L > 0$[p], and $rad(S) \leq L, \forall S \in F$. And, $\forall x \in con(\sum_{S \in F} S), \exists y \in con(\sum_{S \in F} S), \rightarrow |x - y| \leq L\sqrt{N}$.*

Theorem 6. *Caratheodory theorem: $x \in con(A_i, \dots, A_m), A_i \subset R^L, \exists (a_i, a_{m+1}), a_i \in A, x \in con(A_i, \dots, A_{m+1})$.*

Theorem 7. *Shapley-Folkman theorem: $x \in con(A_i, \dots, A_m), A_i \subset R^L, \exists (x = a_i + \dots + a_m), a_i \in conA_i, \forall i, \forall a_i \in A_i$ for $m - i$, (Anderson, Khan et al. 1982)*

Lemma 8. *$x \in con(A_i, \dots, A_m), A_i \subset R^L$:*

$$x = \sum_{i=1}^{m} \sum_{j=0}^{m_j} \lambda_i j a_i j, \lambda_i j > 0, \sum_{i=1}^{m} m_i \leq L, \sum_{j=0}^{m_j} \lambda_i j = 1, \forall i \tag{10}$$

Proof. Proof of Caratheodory theorem : $= 1, x = \sum_{j=1}^{m_1} \lambda_{1j} a_{1j}, m_1 - 1 \leq L, x = x = \sum_{j=1}^{m} \lambda_j a_j, m \leq L + 1$
Proof. Proof of the Shapley-Folkman theorem: $\sum_{i=1}^{m} (m_i - 1) \leq L, m = 1, \nexists (m = 1)$ for $\forall L$ values of i. Now, $a_i = \sum_{j=1}^{m_j} \lambda_{ij} a_{ij} \in conA_i, a_i = \sum_{j=1}^{1} \lambda_{ij} a_{ij} = a_{i1}$.

About the existence of equilibrium in this economy, one can use strict preference relations to prove the existence of such:

1. Commodity space will be given as: R_+^L

[n] Euclidan space that is closed and bounded (all points lie within some fixed distance between each other).
[o] Here C is an arbitrary collection of sets.
[p] Here L is the upper bound for a circle equals $2\pi radd = \pi d$, where d is the diameter. For general shapes, it can be calculate as $\int_0^L ds$.

2. The set of preferences is given as: $\sum_{i=1}^{m} i, \succ_i$ and those preferences satisfy:

   (a) continuity: $\{(x,y) \in R_+^L \times R_+^L : x \succ_i y\}$

   (b) transitivity: $x \succ y, y \succ z, x \succ z$

   (c) Irreflexivity : $:x \not\succ y,$

   (d) Weak monotonicity:$x \gg y, x \succ_i$

If x is in the core of the Edgeworth box,see (Barreto 2009). (exchange economy),$\exists p \in \triangle$:

$$\frac{1}{m}\sum_{i=1}^{m} |p \cdot (x_i - \omega_i)| \leq \frac{2L}{m}\{max||\omega_1||_\infty, \ldots, ||\omega_m||_\infty\} \tag{11}$$

$$\frac{1}{m}\sum_{i=1}^{m} |inf\{p \cdot (y - x_i) : y \succ_i x_i\}| \leq \frac{4L}{m}\{max||\omega_1||_\infty, \ldots, ||\omega_m||_\infty\} \tag{12}$$

In previous expression $||\omega||_\infty$ denotes $\sum_{i=1}^{m} |\omega^i|$ [q], or $||\omega||_\infty = max_i{>}x^i|$ (Anderson 1981). Now if:

$$p \cdot (x_i - \omega_i) = 0 \Rightarrow x_i \in B(p), y \succ_i x_i \Rightarrow p \cdot y \geq p \cdot \omega_i \tag{13}$$

This is Walrasian quasiequilibrum: since agents i consumption need not lie in its budget set $B(p)$, and can be far below the budget frontier. In previous expressions

$$\varphi(p,x,\succ) = |inf{>}p \cdot (y - x_i) : y \succ_i x_i\}| \tag{14}$$

Here $\varphi(p,x,\succ)$ represents a measure that measures how far x is from the demand like (Anderson 1988). If $p \gg 0$, then $\varphi(p,x,\succ) = 0, \exists x \in D(p,(\succ x))$. In the context fo the second welfare theorem a Walrasian quasiequilibrium for an endowment ω, with an income transfer τ is given as: $\sum_{a \in A} f(a) \leq \sum_{a \in A} \omega(a), p \in \triangle$, and $f(a) \in Q(p,a,\tau)$, where $Q(p,a,\tau)$ is a Walrasian quasiequilibria and $Q(p,a,\tau) = Q(\omega,\tau)$. A social - approximate compensated equilibrium(as approximation to market equilibrium) with non-convex preferences consists among other things of modulus A:

$$|(x^1 - y^1) - (x^2 - y^2)| \leq A \tag{15}$$

Also, price vector $p* > 0$, and two allocations $\omega^1 = (x^1, y^1)$ and $\omega^2 = (x^2, y^2)$ (Arrow and Hahn 1971). Now furthermore:

$$|x - y|^2 \leq \sum_{S \in F'} [rad(S)]^2 \leq L^2 = mL^2 \leq nL^2 \tag{16}$$

Here F' is a subfamily of sets of F, m is the number of members of ' , n is the number of commodities or the dimension of space F, L is some number. So:

$$F^{'} \subset F, x \in con \sum_{S \in F'} S, y \in \sum_{S \in F'} S, |x - y| \leq L\sqrt{n} \tag{17}$$

In the previous expression L is some number. The proof of Shapley-Folkman theorem has a lot with Gale-Debreu-Nikaido lemma:

Lemma 9. *Gale-Debreu-Nikaido lemma:*$Z : \triangle \to 2^{R^n}$, *in previous expression* $\triangle$ *is the Laplace operator given by the divergence of gradient(derivative) of a function in a Euclidean space* [r]. *Where* Z *is nonempty, compact, and convex,* $p \in \triangle, \exists z \in Z(p)$, *where* $Z(p)$ *is an excess demand function, so that* $p \cdot z \leq 0$. *And,* $\exists p* \in \triangle, Z(p*) \cap R_-^n \neq \emptyset$, *(Yannelis 1991, Aliprantis, Tourky et al. 2000, Podczeck and Yannelis 2008).*

[q] Actually $\sum_{i=1}^{m} |\omega^i| = \sum_{i \in S} x_i^{\cdot}$, where $x_i^{\cdot} = \omega_i + z_i + \frac{z}{S}$

[r] In Cartesian system Laplace operator is given as the sum of second order partial derivatives of the function with respect to each independent variable.

By the separating hyperplane theorem for $\exists p* \in \triangle, Z(p*) \cap R^n_- \neq \emptyset$, then $\exists q^* \in R^n \setminus 0, upsy \in R^n_- q * \cdot y \leq nfsz \in Z(p) q * \cdot z$. Furthermore $\in \triangle, q_1 \cdot z > 0, q_2 \cdot z > 0, \forall \lambda \in [0,1], \lambda q_1 + (1-\lambda) q_2 \cdot z > 0, q* \in F(p)$, where $F(p), q \in \triangle$ inverse for $F^{-1}(q) = \{p \in \triangle : Z(p) \subset \{z : q \cdot z > 0\}\}$. And, $p* = f(p) \in F(p), p * \cdot z > 0, \exists \forall z \in Z(p*)$, which contradicts Walras law above. Many problems in economics such as existence of competitive equilibrium in general equilibrium theory, can be formulated as fixed-point theorems (Border 1989, Farmakis, Moskowitz et al. 2013). Brouwer theorem states that every continuous function on X has fixed point. The basic Brouwer theorem was set by Brouwer, L. E. J. (Brouwer 1912). Let f be a function mapping of a compact set K in itself. A fixed point of f is a point $z \in K$, satisfying $f(z) = z$, (Border 1989, Shashkin 1991).
FPA asymmetric N-bidder auctions
There is a set of $\Theta = \{1, 2, \ldots, N\}$, of types of bidders. And $\forall \Theta \in \{1, 2, \ldots, N\}$ and $\exists n(\Theta) \geq 1$, which are bidders of type Θ. Bidders of type Θ draw an IPV for the object from CDF $F : [\omega_H, \omega_L] \to R$. It is assumed that $F \in C^2((\omega_H, \omega_L))$ and $f \equiv F^{'} > 0$, on ω_H. The inverse of equilibrium bidding strategy (such as in Maskin and Riley (Maskin and Riley 2000) and Fibich and Gavish (Fibich and Gavish 2011)) is given as :

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b)} = [(\frac{1}{(n-1)} \sum_{j=1}^{n} \frac{1}{v_j(b) - b} - \frac{1}{v_i(b) - b}], i = 1, \ldots, n \tag{18}$$

The initial conditions is given as: $v_i(0) = 0, i = 1, \ldots, n$, the value for the maximum bid is set to: $v_i\{\bar{b}\} = 1, i = 1, \ldots, n$. In the asymmetric case $v_i(b) \neq v(b), v_j(b) \neq v(b)$. In the symmetric case the previous expression would reduce to:

$$b(v_i) = v - \frac{1}{F^{n-1}(v)} \int_r^v {}^{n-1}(s) xs \tag{19}$$

Also

$$\int_r^v {}^{n-1}(s) xs = \int_r^v s^{n-1} ds \tag{20}$$

in general case nonuniform distribution of FPA auction is given as :

$$\beta(v) = x - \int_0^v \frac{F(y)}{F(x)} dy \tag{21}$$

Where $F(y) = 1 - F(x), (F(x)$ is a CDF of a function), x signals are drawn from private values distribution v so $x_i = v_i$. The maximal bid of FPA distributions is then given as

$$\bar{b} = b(1) = 1 - \int_0^1 {}^{n-1}(s) ds \tag{22}$$

The inverse bid functions are solutions of (as in Fibich and Gavious (Fibich and Gavious 2003)):

$$\frac{\partial \cup_i (b; v_i)}{\partial b} = (v_i - b) \sum_{j=1, j\neq 1}^{n} (\prod_{k=1, k\neq 1}^{n} F_k(v_k(b))) z_j(v_j(b)) v_j^{'}(b) - \prod_{j=1, j\neq 1}^{n} {}_j(v_j(b)) = 0 \tag{23}$$

v_i is given fixed, and the maximization problem is:

$$max_b \cup_i (b; v_i) = (v_i - b) \prod_{j=1, j\neq 1}^{n} F_j(v_j(b)), i = 1, \ldots n \tag{24}$$

$$\sum_{j=1, j\neq 1}^{n} \frac{f_j(v_j(b) v_j^{'}(b)}{F_j(v_j(b))} - \frac{1}{v_i(b) - b}, i = 1, \ldots n \tag{25}$$

Or bidder chooses to maximize his expected surplus π_i as in McAfee and McMillan (McAfee and McMillan 1987):

$$\pi_i = (v_i - b_i) F(\beta^{-1}(b_1))^{n-1} \tag{26}$$

And $\frac{\partial \pi_i}{\partial b_i}=0, \frac{dy}{dx}=\frac{\partial \pi_i}{\partial v_i}=F(\beta^{-1}(b_1))^{n-1}$. Another study by Güth, Ivanova-Stenzel, and Wolfstetter (G眉th, Ivanova-Stenzel et al. 2005), uses bid functions for asymmetric bidders proposed by Plum (Plum 1992):

$$b_i^f(v_i)=\omega_l+\frac{v_i-\omega_l}{\sqrt{1+\gamma_i c(v_i-\omega_l)^2}}, \forall i=1,..,n \tag{27}$$

$$c:=\frac{1}{(b_1-\omega_l)^2}-\frac{1}{(b_2-\omega_l)^2} \tag{28}$$

Where $\gamma_i \in [-1,1]$, is the PDF, ω_l is the lower boundary of the statistical distribution (chosen to describe the bidders鈥檅ehavior). In practice bidders valuation are drawn from different statistical distributions, as in Vickrey type auction (Vickrey 1961). In the Vickrey type of auction [s], bidders submit their bids without knowing the other bidders valuations(Vickrey 1961). Vickrey found that in the case where one valuation is commonly known, buyers 1 inverse bid function is given as (Kaplan and Zamir 2012) :

$$v_1(b)=\frac{\beta^2}{4(\beta-b)} \tag{29}$$

This is the case where one valuation is commonly known and where there are two bidders with uniform distributions. Or when $\omega_l=\omega_h=\beta$. If there are 2 bidders only and their values are uniformly and independently distributed on $(0,1)$ and $(0,\omega_h), \omega_h<1$, as in Milgrom (Milgrom 2004):

$$\int_0^{\omega_h} x\cdot\frac{x}{\omega_h}dx+\int_{\omega_h}^1 xdx+\int_0^{\omega_h} y^2dy=\frac{\omega_h^2}{3}+(\frac{1}{2}-\frac{\omega_h^2}{2})+\frac{\omega_h^3}{3}=\frac{1}{2}+\frac{1}{6}\omega_h^2+\frac{1}{3}\omega_h^3 \tag{30}$$

And $E(v_{1,2})=\frac{1}{2}(1-\omega_h)$. In general case as in Plum (Plum 1992), for the probability densities $\varphi_i(x)=c_i(x-\omega_l)^\mu, \omega_l<x<\beta_i, i=1,2\dots n$, there exists and equilibrium solutions:

$$f_1(x)=\omega_l+\frac{1-[1-c(v_i-\omega_l)^k]^{1-\frac{1}{k}}}{c(v_i-\omega_l)^{k-1}}, \omega_l<x\leq\beta_1 \tag{31}$$

$$f_2(x)=\omega_l+\frac{1-[1-c(v_i-\omega_l)^k]^{1-\frac{1}{k}}}{c(v_i-\omega_l)^{k-1}}, \omega_l<x\leq\beta_2 \tag{32}$$

Where $c:=\frac{1}{(b_1-\omega_l)^k}-\frac{1}{(b_2-\omega_l)^k}$; or $c_i:=\frac{\mu+1}{\beta_i-\omega_l)^{\mu+1}}$, where $\mu>-1$, and $k:=((2-\lambda+\mu))/(1-\lambda), \lambda\in[0,1]$. There are k_i - bidders in group i, in total $N=\sum_{i=1}^n k_i$. Bidders submit bids that are solutions to the optimization problem, which is as in Gayle and Richard (Gayle and Richard 2008):

$$\beta(v)=arg\,max_{u\in(0,\omega_h)}(v-u)\cdot[F_i(\lambda_i(u))]^{k_i-1}\prod_{j\neq 1}[F_j(\lambda_j(u))]^{k_j} \tag{33}$$

$\exists u=\sum_{i=1}^n u_i$, where u_i denotes the player of type i. Truncated CDF in general form is given as:

$$F^*(v)=\prod_{j=1}^n\frac{F_j(v)-F_j(\omega_l)}{F_j(\omega_h)-F_j(\omega_l)} \tag{34}$$

Probabilities of winning the auction are given by the following expression:

$$p_i(r)=k_i\int_r^{b(\omega_h)}\frac{l_i'(v)}{l_i(v)}\prod_{j=1}^n[l_j(v)]^{k_j}dv \tag{35}$$

[s] In Vickrey type of auction each bidder bids its own valuation, and this is optimal strategy. This is a sealed bid auction where the highest bid wins, but pays only the second highest bid.

Where $l_i(v) = F_i(\lambda_i(v))$. Also, r represents the reserve price in auction. Expected revenue for the auctioneer is given by:

$$E(p,b_i,v_i) = \omega_h - r\prod_{j=1}^{n}[F_j(r)]^{k_j} - \int_{r}^{b(\omega_h)} \frac{l_i'(v)}{l_i(v)} \prod_{j=1}^{n}[l_j(v)]^{k_j} dv \tag{36}$$

Group i bidders expected revenue is given by:

$$E_i(p,b_i,v_i) = k_i \int_{r}^{b(\omega_h)} [F_i^{-1} l_i(v)) - v] \cdot \frac{l_i'(v)}{(l_i(v)} \prod_{j=1}^{n}[l_j(v)]^{k_j} dv \tag{37}$$

Now if $U(p_i,E_i,r) = p_i \cdot (r - E_i)$, by the envelope theorem optimal values are denoted by asterisk $^{*}r'(r) = p^{*}(r)$, as in Milgrom (Milgrom 1989), and one can integrate to obtain the previous result.

$$U^{*}(x) = \int_{0}^{r} p^{*}(v)dv \tag{38}$$

Previous proof confirms the Revenue equivalence theorem. Revenue equivalence theorem confirms that if there are n risk neutral agents, that do independent and personal evaluation of some auction good, and valuation follows cumulative distribution $F(v)$, which is ascending probability distribution of a continuous set of choices $(v,\overline{v})$. Than every auction mechanism (every institution auction), in which lot will be allocated towards the agent for which it has highest value $\overline{v}$, and every agent with a valuation of good v has utility 0, generates exact same revenue, which lead every bidder to make the same payment. FPA sealed-bid, SPA sealed-bid auctions generate the same price on average. This result is confirmed in: Vickrey (Vickrey 1961), Ortega-Reichert (Ortega-Reichert 1967), Myerson (Myerson 1981), Riley and Samuelson (Riley and Samuelson 1981), etc. At BNE each player tries to maximize its own expected payoff $E(v_i - b)||(\beta_{-i} < b_i)|v_1$, as in Campo, Perrigne and Vuong (Campo, Perrigne et al. 2003). Now, $\beta_{-i} = max\{v_1(y_{1i}^{*}), v_0(y_{0i}^{*})\}, y_{1i}^{*} = max_{j \neq i, j \in G_1} v_{i=1,j}$, where G_1 is a cartel of better informed bidders and $y_{0i}^{*} = max_{j \neq i, j \in G_0} v_{i=0,j}, G_0$ are weak i.e. less informed bidders. And v_1 and v_0 are the equilibrium bids of the bidder types 1 and 0 respectively. Maximization problem for any bidder can be written as:

$$max_{b_{1i}} = (v_{1i} - b_{1i})Prob(y_{1i}^{*} \leq v_1^{-1}(b_{1i})) \wedge y_{0i} \leq v_0^{-1}(b_{1i})|v_{1i} \tag{39}$$

Probability can be written as: $F_{y_1^{*},y_0|v_1}(v_1^{-1}(b_{1i})), v_0^{-1}(b_{1i})|v_{1i}$. FONC for v_{1i} bidder is given as:

$$-F_{y_1^{*},y_0|v_1}(v_1^{-1}(b_{1i})), v_0^{-1}(b_{1i})|v_{1i} +$$

$$\begin{aligned}(v_{1i} - b_{1i})[&\frac{(\partial F_{y_1^{*},y_0|v_1}(v_1^{-1}(b_{1i})), v_0^{-1}(b_{1i})|v_{1i}}{\partial y_{1i}^{*}} \cdot \frac{1}{v_1'(v_1^{-1}(b_{1i}))} \\ &+ \frac{(\partial F_{y_1^{*},y_0|v_1}(v_1^{-1}(b_{1i})), v_0^{-1}(b_{1i})|v_{1i}}{\partial y_0} \cdot \frac{1}{(v_0'(v_0^{-1}(b_{1i}))}]\end{aligned} \tag{40}$$

In the previous expression $v_{1i} \in [\omega_l, \omega_h]$, and $b_{1i} = v_1(v_{1i})$. FONC for v_{0i} bidder can be obtained in the similar manner. A market action should be attainable by actions of a consumer or a collation of consumers. This leads us to the concept of core economy Ce. Every allocation in the core of the economy is said to be Pareto efficient, i.e. the allocation cannot be changed such that one agent is strictly better off without making any other agent worse off. And furthermore, the set of all competitive equilibrium allocations $C\varepsilon$ is contained in the core, $C\varepsilon \subset Ce$ (Jain 2004). The question that arises here is whether core of the economy is equivalent to the competitive equilibrium allocations. Shapley and Shubik studied one asymmetric economy in the indivisible goods markets (houses) but with only one or two buyers (Shapley and Shubik 1971). These are non-combinatorial market cases. One example of combinatorial market is given in Bikhchandani and Mamer (Bikhchandani and Mamer 1997). Early attempt to explain indivisible goods market by "matching models" (college admissions and university quotas) and "stable marriage" assignment problem were analyzed. These papers proved that indivisible goods market economy has non-empty core. Knaster-Kuratowski-Mazurkiewicz lemma is a basic result in fixed point theory. Or Knaster-Kuratowski-Mazurkiewicz-Shapley theorem which is a generalization of previous lemma, based on Brower fixed point theorem. Theorem K-K-M-S is stated as follows: $C_s : S \subset N$ is a family of closed subsets of simplex Δ^N. Where $\Delta^N = \{x \in R^n : x_i > 0, \wedge \sum_{i=1}^n x_i = 1\}$, is a $n-1$ simple, for $S \subset N, \Delta^S = \{x \in \Delta^n : x_i > 0, \wedge \sum_{i \in S} x_i = 1\}$. Now, we can assume that $\Delta^T \subset \cup_{S \subset T\, s}, \forall T \subset N, \exists B, \cap_{s \in B} C_s \neq \emptyset$. Where, B is a balanced family (collection), such that intersection of sets indexed by B is nonempty (Krasa and Yannelis 1994) [t].And "the core approaches the set of equilibrium allocations as the number of traders tends to infinity", and a continuum economy is an appropriate mathematical model of a situation of existence of "many" commodities (Aumann 1964).

Literature review on previous research concerning numerical solutions on asymmetric auctions

The first researchers to propose using numerical algorithms to solve for the equilibrium or inverse bid functions were Marshall, Meurer, Richard and Stromquist (Marshall, Meurer et al. 1994). They applied l'Hopital's rule to the FOC to derive: $\lim_{s \to 0^+} \to \varphi_k^{'}(b) = \frac{(n_k+1)}{n_k}$. Where $\varphi_k^{'}(b)$ is a first derivative of the inverse bid function, and n_k are the number of bidders in coalition. Inverse bid functions are in practice normalized to $\delta_k(b) = \frac{\varphi_k(b)}{b}$. They approximated $\{\delta_k(b)\}_k^2 = 1$ by Taylor series expansions of order 5 around each point $b_t \in [\omega_L, \omega_H]$. Condition for valid solution is: $\frac{1}{2}\sum_{k=1}^2 [\delta_k(\omega_L) - \frac{(n_k+1)}{n_k}] \leq \varepsilon^2$, where ε is some tolerance level (convergence criterion), of order $10^{-5} - 10^{-8}$. Bajari (2001) proposed that inverse bid function (n-bidders) can be represented as a linear combination of ordinary polynomials (Bajari 2001). This can be presented in the following manner:

$$\widehat{\omega}_n(b) = \overline{b} - \sum_{k=0}^{K} \alpha_{n,k} (\overline{b} - b)^k, n = 1, 2, \ldots, N \tag{41}$$

FOC for bidder n can be expressed as:

$$1 = [\varphi_n(b) - b] \sum_{m \neq n} \frac{f_m[\varphi_m(b)]}{F_m[\varphi_m(b)]} \varphi_m'(b) \tag{42}$$

and

$$G_n(\omega_L, \omega_H, \alpha) = 1 - [\varphi_n(b) - b] \sum_{m \neq n} \frac{f_m[\varphi_m(b)]}{F_m[\varphi_m(b)]} \varphi_m(b) \tag{43}$$

And the left boundary condition is: $\varphi_n(\omega_L) = \omega_L$, and the right boundary condition is $\varphi_n(\omega_H) = \omega_H$. In Fibich and Gavious (Fibich and Gavious 2003), is suggested using a perturbation analysis to calculate an explicit approximation to the asymmetric FPA solution, they defined average distribution between N bidders at a valuation v, namely : $F_{average} \equiv \frac{1}{N}\sum_{n=1}^N F_n(v)$. The parameter that measures the level of asymmetry is given as: $\varepsilon = max_{n \in \{1,\ldots,N\}} > ax_{v \in \{\omega_L, \omega_H\}} > F_n(v) - F_{average}(v)|$, and that $F_n(v) = F_{average}(v) + \varepsilon A_n(v), n = 1, ..N$, where auxiliary function $A_n(v) = \frac{F_n(v) - F_{average}(v)}{\varepsilon}$. Equilibrium bid function [u], when bidders draw valuations from $F_{average}(\cdot)$ is given (Schmedders and Judd 2013):

$$E_n[v, A_n(v)] = \frac{1-N}{F_{average}^{N-1}(v)}.$$

[t] In the proof of this theorem provided in this paper Krasa and Yannelis (1994) it is defined set valued function: $\psi = \Delta^N \to 2^{R^n}, F(x) = con999$, where $m^S = \{m_1^S, \ldots, m_N^S\}$ is the center of the simplex Δ^S.And $m_i^S = \frac{1}{|S|}, i \in S, m_i^S = 0, i \notin S$.And $\psi(x) = \{y : f(x)x > f(x)y\}, \exists T \subset N$, and $int\Delta^n = \{x = (x_0, \ldots, x_n) \in R^{n+1} | \sum_0^n x_i = 1, x_i > 0, \forall i\}$, therefore since $S \subset T, S \in I(x^*)$ and $m^S \in F(x^*)$. And the $f(x^*)x^* = f(x)m^S > f(x^*)m^N$.

[u] Initial guess about maximum bid function is given as: $\overline{b} = \omega_H - \int_{\omega_L}^{\omega_H} F_{average}^{N-1}(v)dv$

$$\cdot [\int_{\omega_L}^{v} {}^{N-1}_{average}(v) dv q^N \int_{v}^{\omega_H} \frac{1}{[F_{average}^{N-1}(v)]^{N-1}} \frac{d[\frac{A_n(t)}{(F_{average}(t))}]}{dt} dt \tag{44}$$

Gayle and Richard (Gayle and Richard 2008), generalized backward shooting algorithm, they defined: $l_n(v) \equiv F_n[\varphi_n(v)]$, and now FOC can be defined as: $1 = (F_n^{-1}[l_n(v)] - v) \sum_{m \neq n} \frac{l_n'(v)}{l_n(v)}$, and the high bid is chosen by solving:

$$min_{b[r,\omega_H]} \sum_{n=1}^{N} [l(r|b^-) - F_n(r)]^2, r$$

is reserve price. Hubbard and Paarsch (Hubbard and Paarsch 2009) used Chebyshev polynomials, which are orthogonal polynomials and more stable. Chebyshev nodes can be computed as: $x_t = cos[\frac{\pi(t-1)}{T}], t = 1, \ldots, T$. The points $\{v_t\}_t^T = 1$ are found via transformation like this: $v_t = \frac{\overline{b} + \omega_L + (\overline{b} - \omega_L)x_t}{2}$. Chebyshev polynomials can be defined recursively as $T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) + T_{n-1}(x)$. The coefficients of these polynomials for a function $f(x)$ can be obtained by the following integral:

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_n(x)}{(1-x^2)^{1/2}} dx \tag{45}$$

Hubbard, Kirkegaard and Paarsch imposed 5 in/equality constraints on the equilibrium bid functions [v], that are approximated by the Chebyshev polynomials of orderK (Hubbard, Kirkegaard et al. 2013). They approximated the solution to differential equations (general FPA model):

$$\varphi_n^{'}(v) = \frac{F_n[\varphi(v)]}{f_n[\varphi(v)]} ([\frac{1}{N-1} \sum_{m=1}^{N} \frac{1}{\varphi_n(v) - v}] - \frac{1}{\varphi_n(v) - v}) \tag{46}$$

And they are doing so by minimizing and solving:

$$min_{\overline{b},\alpha} \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(\omega_L, \omega_H, \alpha)]^2.$$

Where in previous minimization problem

$$G_n(\omega_L, \omega_H, \alpha) = 1 - [\varphi_n(b) - b] \sum_{m \neq n} \frac{f_m[\varphi_m(b)]}{(F_m[\varphi_m(b)]} \varphi_m'(b).$$

Kirkegaard proved that if the CDFs cross each other ($F_n(v)$ crosses $F_m(v)$), than their bid functions would cross each other. Relative power of n bidder over m bidder and the utility of bidder n is measured as (Kirkegaard 2009):

$$P_{n,m}(v) = \frac{F_m(v)}{F_n(v)}, v \in (\omega_L, \omega_H) \, and \, U_n(v) = (v - \omega_L) \prod_{m \neq n} F_m[\varphi(\omega_L)] \tag{47}$$

And the ratio of n's equilibrium pay-off relative to m's equilibrium pay-off is: $R_{n,m}(v) = \frac{U_n(v)}{U_m(v)}, v \in (\omega_L, \omega_H)$. Kirkegaard assumes that comparing two ratios $\forall v \in (\omega_L, \omega_H)$, is equivalent of comparing two bids $b_n(v)$ and $b_m(v)$, i.e.: $R_{n,m}(v) \geq P_{n,m}(v) \Leftrightarrow b_n(v) \geq b_m(v)$. Right boundary condition is : $R_{n,m}(\omega_H) = P_{n,m}(\omega_H) = 1$, left boundary condition is : $lim_{b \to \omega_L} R_{n,m}(b) = \frac{f_m(\omega_L)}{f_n(\omega_L)} = lim_{b \to \omega_L} P_{n,m}(b)$ (Kirkegaard 2009).

Asymmetric auctions: simulation results

In this part we choose 10 bidder types, there is only one bidder from each type, and these bidders draw their

[v] 1. $\varphi_n(v) = \omega_L$, 2.$\varphi_n(\overline{b}) = \omega_H$ 3.$\sum_{m \neq n}(\overline{b} - \overline{v}) f_m(\overline{b}) \varphi_m^{'}(v) = 1$, 4.$\varphi(\omega_L) = \frac{N-1}{N}$, 5.$\varphi_n(v_j - 1) \leq \varphi_n(v_j)$, for some uniform array $j = 2, \ldots, J$.

IPVs from for the object of the auction from their CDF $F : [\omega_H, \omega_L] \to R$. Ten selected distributions in the following order are, (see, (Johnson, Kemp et al. 2005)):

Table 1 Selected distributions and their CDFs and PDFs

Distributions and boundaries	**CDF**	**PDF**
Beta $[0,1]$	$F(x)=\frac{1}{B(a,b)}\int_{\omega_L}^{\upsilon(x)} x^{a-1}*(1-x)^{b-1}dx;$ $\upsilon(x)=\frac{x-\omega_L}{\omega_H-\omega_L}$	$f(x)=\frac{1}{\omega_H-\omega_L}\frac{\upsilon(x)^{a-1}(1-\nu(x))^{b-1}}{B(a,b)}$
Exponential $[0,1]$	$F(x)=\frac{1-\exp(-\lambda(x-\omega_L)}{1-\exp(-\lambda(\omega_H-\omega_L)}$	$f(x)=\frac{\lambda\exp(-\lambda(x-\omega_L)}{1-\exp(-\lambda(\omega_H-\omega_L)}$
Gamma $[0,1]$	$F(x)=\frac{\int_0^\infty x^{k-1}e^{-x}dx}{\Gamma(k)}$	$f(x)=\frac{1}{\theta^k\Gamma(k)}x^{k-1}e^{-\frac{x}{\theta}}$
Kumaraswamy $[0,1]$	$F(x;a;b)=1-(1-x^a)^b$	$f(x;a,b)=F'(x;a;b)=abx^{a-1}(1-x^a)^{b-1}$
Lognormal $[0,1]$	$F(x)=\frac{\int_a^x \frac{1}{z\sigma\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{lnz-\mu}{\sigma}\right)^2\right]dz}{\int_a^b \frac{1}{z\sigma\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{lnz-\mu}{\sigma}\right)^2\right]dz}$	$f(x)=\frac{\frac{1}{x\sigma\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{lnx-\mu}{\sigma}\right)^2\right]}{\int_a^b \frac{1}{z\sigma\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{lnz-\mu}{\sigma}\right)^2\right]dz}$
Standard normal $[0,1]$	$F(x)=\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}$	$f(x)=\frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{\omega-\mu}{\sigma}\right)-\Phi\left(\frac{-\mu}{\sigma}\right)}$
Power $[0,1]$	$F(x)=\frac{\eta}{\alpha+1}\left[(x+a+c)^{\alpha+1}-c^{a\mp1}\right]$	$f(x)=\eta(x+a+c)^{\alpha}.$ $\eta=(\alpha+1)\left[(x-a+c)^{\alpha+1}-c^{a+1}\right]^{-1}$
Reverse power $[0,1]$	$F(x)=1-\left(\frac{b-x}{b-a}\right)^{\alpha}$	$f(x)=\frac{\alpha(b-x)^{\alpha-1}}{b-a}$
Triangular $[0,1]$	$F(X)=\psi x+(1-\psi)\frac{(x-a)^2}{(b-a)(c-a)}$	$f(x)=F'(x)=\begin{cases}4x\ if\ x\in(0,\ldots,0,5)\\ 4-4xif\ \ x\in(0.5,\ldots,1)\end{cases}$
Uniform $[0,1]$	$F(x)=\frac{x-\omega_L}{\omega_H-\omega_L}$	$f(x)=\frac{1}{\omega_H-\omega_L}$

Java software has been applied for the simulations. The software name is Auction Solver and was written by Richard M. Katzwer, see (Katzwer 2012). Next, in a table 2 are presented Chebyshev coefficients for 10 parametrized distributions with their CDFs and PDFs written in Table 1.

Table 2 Solution: Chebyshev coefficients of K=15 degree

Beta distribution	Exponential distribution	Gamma distribution	Kumaraswamy distribution	Log nomal distribution	Standard normal distribution	Power l distribution	Reverse power distribution	Triangular distribution	Uniform distribution
0.0020	0.0031	0.0021	0.0020	0.0063	0.0032	0.0031	0.0011	0.0028	0.0033
0.9845	0.9827	0.9813	0.9845	0.9886	0.9817	0.9807	0.9973	0.9823	0.9831
0.0024	0.0035	0.0025	0.0024	0.0061	0.0036	0.0034	-0.0004	0.0028	0.0038
0.0080	0.0104	0.0112	0.0080	0.0037	0.0103	0.0111	0.0144	0.0088	0.0100
0.0000	0.0000	0.0000	0.0000	-0.0006	0.0000	0.0000	-0.0006	-0.0003	0.0000
-0.0011	-0.0003	0.0005	-0.0011	-0.0010	0.0001	0.0005	-0.0066	0.0010	-0.0009
0.0000	0.0000	0.0000	0.0000	-0.0004	0.0000	0.0000	-0.0003	0.0000	0.0000
0.0008	0.0007	0.0008	0.0008	-0.0018	0.0007	0.0009	-0.0048	0.0014	0.0004
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0014	0.0000	0.0000
0.0004	-0.0001	0.0000	0.0004	-0.0007	0.0000	0.0000	-0.0025	-0.0006	0.0001
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0007	0.0000	0.0001	0.0007	-0.0001	0.0003	0.0001	-0.0004	0.0001	0.0002
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
High bid: $\overline{\boldsymbol{b}} = 0.893113927397631$									

Table 2 Solution: Chebyshev coeff. of K=15 order reserve price =0.5

Beta distribution	Exponential distribution	Gamma distribution	Kumaraswamy distribution	Log nomal distribution	Standard normal distribution	Power l distribution	Reverse power distribution	Triangular distribution	Uniform distribution
0.4432	0.4224	0.4378	0.3424	0.2980	0.3970	0.4438	0.3830	0.4010	0.4123
0.6213	0.6570	0.6284	0.7916	0.8710	0.7012	0.6191	0.7359	0.6901	0.6757
-0.0764	-0.1015	-0.0877	-0.1854	-0.2211	-0.1301	-0.0803	-0.0853	-0.1231	-0.1121
0.0190	-0.1015	0.0298	0.0611	0.0615	0.0447	0.0255	-0.0338	0.0369	0.0358
0.0013	-0.0031	-0.0049	-0.0073	0.0029	-0.0064	-0.0032	0.0000	-0.0027	-0.0035
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
High bid: $\overline{\boldsymbol{b}} = 0.9032115516025931$									

Table 3 Backward shooting method solution, end result: Convergence true

	b_bar	A	B	B-A	Result	
0/39	0.5	0	1	1.00E+00	3	Solution not within specified tolerances.
1/39	0.75	0.5	1	5.00E-01	3	Solution not within specified tolerances.
2/39	0.875	0.75	1	2.50E-01	2	Solution diverges to +infinity.
3/39	0.8125	0.75	0.875	1.25E-01	2	Solution diverges to +infinity.
4/39	0.78125	0.75	0.8125	6.25E-02	3	Solution not within specified tolerances.
5/39	0.7968750	0.7812500	0.8125000	3.13E-02	3	Solution not within specified tolerances.
6/39	0.8046875	0.7968750	0.8125000	1.56E-02	3	Solution not within specified tolerances.
7/39	0.8085938	0.8046875	0.8125000	7.81E-03	3	Solution not within specified tolerances.
8/39	0.8105469	0.8085938	0.8125000	3.91E-03	2	Solution diverges to +infinity.
9/39	0.8095703	0.8085938	0.8105469	1.95E-03	3	Solution not within specified tolerances.
10/39	0.8100586	0.8095703	0.8105469	9.77E-04	3	Solution not within specified tolerances.
11/39	0.8103027	0.8100586	0.8105469	4.88E-04	3	Solution not within specified tolerances.
12/39	0.8104248	0.8103027	0.8105469	2.44E-04	2	Solution diverges to +infinity.
13/39	0.8103638	0.8103027	0.8104248	1.22E-04	2	Solution diverges to +infinity.
14/39	0.8103333	0.8103027	0.8103638	6.10E-05	3	Solution not within specified tolerances.
15/39	0.8103485	0.8103333	0.8103638	3.05E-05	1	Solution diverges to -infinity.
16/39	0.8103409	0.8103333	0.8103485	1.53E-05	3	Solution not within specified tolerances.
17/39	0.8103447	0.8103409	0.8103485	7.63E-06	2	Solution diverges to +infinity.
18/39+	0.8103428	0.8103409	0.8103447	3.82E-06	3	Solution not within specified tolerances.
19/39	0.8103437	0.8103428	0.8103447	1.91E-06	2	Solution diverges to +infinity.
20/39	0.8103433	0.8103428	0.8103437	9.54E-07	2	Solution diverges to +infinity.
21/39	0.8103430	0.8103428	0.8103433	4.77E-07	3	Solution not within specified tolerances.
22/39	0.8103431	0.8103430	0.8103433	2.38E-07	2	Solution diverges to +infinity.
23/39	0.8103431	0.8103430	0.8103431	1.19E-07	3	Solution not within specified tolerances.
24/39	0.8103431	0.8103431	0.8103431	5.96E-08	3	Solution not within specified tolerances.
25/39	0.8103431	0.8103431	0.8103431	2.98E-08	3	Solution not within specified tolerances.
26/39	0.8103431	0.8103431	0.8103431	1.49E-08	3	Solution not within specified tolerances.
27/39	0.8103431	0.8103431	0.8103431	7.45E-09	2	Solution diverges to +infinity.
28/39	0.8103431	0.8103431	0.8103431	3.73E-09	2	Solution diverges to +infinity.
29/39	0.8103431	0.8103431	0.8103431	1.86E-09	3	Solution not within specified tolerances.
30/39	0.8103431	0.8103431	0.8103431	9.31E-10	2	Solution diverges to +infinity.
31/39	0.8103431	0.8103431	0.8103431	4.66E-10	3	Solution not within specified tolerances.
32/39	0.8103431	0.8103431	0.8103431	2.33E-10	2	Solution diverges to +infinity.
33/39	0.8103431	0.8103431	0.8103431	1.16E-10	3	Solution not within specified tolerances.
34/39	0.8103431	0.8103431	0.8103431	5.82E-11	3	Solution not within specified tolerances.
35/39	0.8103431	0.8103431	0.8103431	2.91E-11	2	Solution diverges to +infinity.
36/39	0.8103431	0.8103431	0.8103431	1.46E-11	3	Solution not within specified tolerances.
37/39	0.8103431	0.8103431	0.8103431	7.28E-12	2	Solution diverges to +infinity.
38/39	0.8103431	0.8103431	0.8103431	3.64E-12	2	Solution diverges to +infinity.
39/39	0.8103431	0.8103431	0.8103431	1.82E-12	3	Solution not within specified tolerances.
Highest bid: $\bar{b} = 0.810343140133682$. Shooting terminated at $b = 0.5000231401337242$. (b_underbar = 0.5)						

Next in Table 4 are presented parameters for the Constrained strategic equilibrium and Backward shooting solver.

Table 4 C.S.E. and Backward shooting parameters

Constrained strategic equilibrium (no reserve price) parameters		**Constrained strategic equilibrium (reserve price = 0.5) parameters**		Backwards solver parameters	
T (degree)	40	T (degree)	40	Shooting method	Euler
K (grid)	15	K (grid)	15	ODE system	Inverse bid functions
μ_l	2000	μ_l	2000	h1 (tolerance of the deviation of the solution from left boundary)	1.0E-5
μ_h	5000	μ_h	5000	h2 (step size close to high bid)	0.001
μ_{FONC}	5.0	μ_{FONC}	5.0	Threshold	0.01
$\mu_{\bar{b}}$	0.0	$\mu_{\bar{b}}$	0.0	High-bid precision	1.0E-12
μ_{mono}	1000	μ_{mono}	1000	Left-boundary tolerance	1.0E-5
Cheb grid	= no	Cheb Grid	= no	/	/

Euler method used in backward shooting solver is described as the simplest Runge-Kutta method, ODE is of the form: $\frac{dy(t)}{dt} = f(t,y(t)), y(t_0) = y_0, \frac{dy(t)}{dt} \approx \frac{y(t+h)-y(t)}{h}, y(t+h) \approx y(t) + h\frac{dy}{dt}$. The iterative solutions is than given as: $y_n + 1 = y_n + hf(t_n, y_n)$ or in previous expressions $x \in (x_0, x_n)$. MATLAB also is a powerful tool used by economists and can compute equilibrium strategies in a first-price auction with two players using the boundary value method with fixed-point iterations, and by using the boundary value method with Newtons iterations. A fixed point is a point that does not change upon of a function (map), system of differential equations etc. (Shashkin 1991). In the Newton's method the algorithm can be applied iteratively to obtain: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n-1})}$, if $\lim_{x_{n+1}\to x^*} \frac{f(x_n)}{f'(x_n)} = x_n$, and $x_n = x^* + \varepsilon_n$, where $\varepsilon_{n+1} = \frac{f''(x^*)}{2\cdot f'(x^*)}\varepsilon_n^2$. Fixed point theorem states that if $\exists f(x) \in [a,b]$, then $\exists x \in [a,b]$, and $f(x) - x = 0 \Rightarrow f(x) = x$, see (Rosenlicht 1968). In our case Newton's method has quadratic convergence, i.e. we can denote residual (in two bidders case as, (see (Fibich and Gavious 2003, Fibich and Gavish 2011)):

$$\varepsilon_b[b', v_1, b, v_2] = b'(v_2) - \frac{f_2(v_2)}{F_2(v_2)}(v_1 - b) \tag{48}$$

In a two bidders case one follows power law distributions, with CDF, $F_1 = c_1 v_1^{a_1}$. And the second bidder distribution is truncated normal with CDF: $F_2 = c_2 \cdot erf(\frac{a_{f_2}}{a_{f_2+1}} \cdot \frac{v}{\sqrt{2}})$, where the error return function is defined as:

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

see (Abramowitz and Stegun 1964). Matlab code for this simple two bidder case was written by (Fibich and Gavish 2011). In the next two graphs are presented two bidder's distribution valuations.

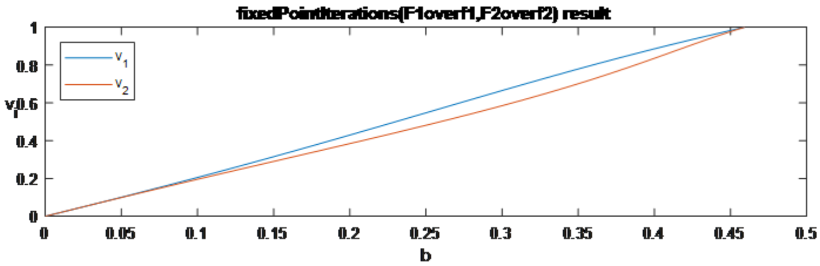

Fig. 1 Fixed point iterations result of the ratios of the two bidders' valuations CDF/PDF functions

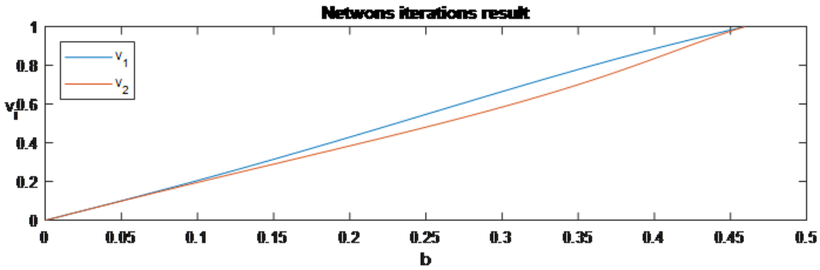

Fig. 2 Newtons iterations result of the two CDF/PDF bidders' valuations functions

5 Conclusion

As it is known competitive equilibrium need not exist in the indivisible goods economy. These markets include auctions where discrete items can be only traded, as a whole only. In combinatorial auction there is a finite set of times but one buyer can only buy subset of items. All the previous research on the topic studied the competitive equilibrium where there are only one or two buyers or sellers (Shapley and Shubik 1971, Khan and Rath 2013). These markets are non-combinatorial since only one item (commodity) at time is a subject of sale. The attempt to deal with non-convex preferences in finite setting by proposing approximate equilibria (Starr 1969, Starr 2011, Starr 2016).
This paper was written by following of this idea. In this paper asymmetric auction was used in order to prove the previous theoretical models. Auction in most general terms is a game theoretic mechanism which allocates an object (set of objects) and is composed of set of bidders χ, set of objects allocated , a private type space Θ, and public type space Ξ. And where each bidder has type of distributions $\{\theta_i, \xi_i\} \in \Theta \times \Xi$, and $\Theta \times \Xi = \sum_{i=1}^{N} \Theta_i \times \sum_{i=1}^{N} \Xi_i$, which represents the space of all type profiles, see (Katzwer 2012). In the FPA auction that was used for analysis in this paper every bidder pays its bid, the bidder does not know the opponents' bids. In the asymmetric type of FPA auctions highest bid wins, and highest winning bidder pays its bid. So, the highest bid is considered to be the equilibrium bid.
In this type of market setting contrary to the convex case, where agents with convex preferences that do not prefer extremes and they prefer in between values, in the First price auction BNE equilibrium is bidders i bid and that must be highest bid so that the item is allocated to him, and the outcome is efficient. Since in the First price items is sold to the buyer with the highest valuation of the item, this auction mechanism is Pareto efficient. Though in theory FPA auction is distinct from the English type of auction since here bidders can only submit one, bid and they do not know other bidders valuation and they may bid too low.

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