COMPOSITIONS INVOLVING DIRAC-DELTA AND HEAVISIDE FUNCTIONS

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NEUTRIX CALCULUS

- 1932, Jacques Hadamard

$$\int_0^1 \frac{A(x)}{x^{p+1/2}} dx, \tag{1}$$

$p \in \mathbb{Z}^+$, $A(x)$ is infinitely differentiable function.

$$\int_\epsilon^1 \frac{A(x)}{x^{p+1/2}} dx = F(\epsilon) + I(\epsilon),$$

$$F(\epsilon) = \int_\epsilon^1 \frac{A(x) - B(x)}{x^{p+1/2}} dx, \; I(\epsilon) = \int_\epsilon^1 \frac{B(x)}{x^{p+1/2}} dx, \; B(x) = \sum_{i=0}^{p-1} \frac{A^{(i)}(0)}{i!} x^i.$$

NEUTRIX CALCULUS

Van der Corput has used the Hadamard finite part in his asymptotic researches.
In 1959 Van der Corput in **Introduction to the neutrix calculus** has established the Neutrix calculus.

-In the 80's in the previous century, **Brian Fisher** has extended the definitions about products, compositions and covolution products to larger class of distributions by using of the Neutrix calculus.

Application of the Neutrix calculus

- In the theory of quantum fields in order to obtain finite results for the coeffficients in perturbation series, by Jack Ng and van Dam.
- Sawomir Sorek has used the neutrix product of the distributions in non-linear systems in electronics, signal proccessing and in the telecomunications.

NEUTRIX CALCULUS

By **van der Corput**:

Definition

Let N' be a set and let N be a commutative, additive group of functions mapping N' into a commutative, additive group N''. If N has the property that the only constant function in N is the zero function, then N is said to be a neutrix and the functions in N are said to be negligible.
The condition that the constant function in N is only zero function is called neutrix condition.

NEUTRIX CALCULUS

Definition

Let $f(\xi)$ be a real (or complex) valued function defined on N' and suppose it is possible to find a constant l such that $f(\xi) - l$ is negligible in N. Then l is called the neutrix limit or N-limit of $f(\xi)$ as ξ tends to b and we write

$$N - \lim_{\xi \to b} f(\xi) = l.$$

If the neutrix limit l exists, then it is only one.

Neutrix compositions of distributions

In the Schwartz' theory of distributions:

$F(f(x)) = ?$, when F and f are arbitrary distributions.

Example: $\delta^2 = ?, \sqrt{\delta} = ?$

Antosik and Fisher have made many tries in order to extend the definitions for composition of distribution F and locally integrable function f in the same way like composition of two distributions is defined.

Neutrix compositions of distributionsi

Antosik:
The regular Temple's sequences are defined δ-sequences $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$ and they converge to Dirac δ function (as distribution).

$\rho(x)$ is fixed infinitely differentiable function on $\mathbb{R}$, with the following properties:

(1) $\rho(x) = 0$ за $|x| \geq 1$,

(2) $\rho(x) \geq 0$,

(3) $\rho(x) = \rho(-x)$,

(4) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Neutrix compositions of distributions

Let F is distribution in $\mathcal{D}'$ and if $F_n(x) = \langle F(x-t), \delta_n(x)\rangle$, then $F_n(x)$ is regular sequence of infinitely differentiable functions which converges to $F(x)$.

Definition

Let $f, g \in \mathcal{D}'$. We said that the distribution $g(f(x))$ exists and it is equal to $h(x)$ on $\mathbb{R}$ if the sequence from compositions $\{g_n(f_n)\}$ converge to the distribution $h(x)$.

Neutrix compositions of distributions

Many compositions of distributions are calculated by using of the previous definition:

(i) $\sqrt{\delta} = 0$

(ii) $\sqrt{\delta^2 + 1} = 1 + \delta$

(iii) $\log(1 + \delta) = 0$

(iv) $\sin\delta = 0$

(v) $\cos\delta = 1$

(vi) $\frac{1}{1+\delta} = 1.$

Neutrix compositions of distributions

We used the following definition:

Definition

Let F is a distribution in $\mathcal{D}'$ and let f is locally integrable function. It is said that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N - \lim_{n\to\infty} \int_{-\infty}^{+\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle \tag{2}$$

for all test functions φ vo $\mathcal{D}[a, b]$, where N is a neutrix with domain $N' = \mathbb{Z}^+$, range $N'' = \mathbb{R}$ and negligible functions

$$n^\lambda \ln^{r-1} n, \ln^r n : \lambda > 0, r = 1, 2, \dots \tag{3}$$

and all functions which converge to 0 in the usual sense, when n tends to the infinity.

Neutrix compositions of particular distributions

1. **L.Lazarova**, B.Jolevska-Tuneska, I.Akturk, E.Ozcag, *Note on the distribution composition* $(x_+^\mu)^\lambda$, Bulletin of the Malaysian Mathematical Sciences Society, Springer, (2016), pp.1-13
2. E.Ozcag, **L.Lazarova**, B. Jolevska-Tuneska, *Defining compositions of* $x_+^\mu, |x|^\mu, x^{-s}$ *and* $x^{-s}\ln|x|$ *as a neutrix limit of regular sequences*, Communications in Mathematics and Statistics, Volume 4, Issue 1, Springer Berlin Heidelberg, (2016): pp. 63-80.

Neutrix composition of the distributions x^λ and x_+^μ for $\lambda = -1, -2, \ldots, \mu > 0$ and $\lambda\mu \in \mathbb{Z}^-$

We used the Fisher principe in relation to calculate the compositions of the distributions.

The local integrable functions $x_+^\lambda, x_-^\lambda, |x|^\lambda$ for $\lambda > -1$ are defined with:

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0 \\ 0, & x < 0 \end{cases}, \quad x_-^\lambda = \begin{cases} |x|^\lambda, & x < 0 \\ 0, & x > 0 \end{cases}, \tag{4}$$

and $|x|^\lambda = x_+^\lambda + x_-^\lambda$.

The distributions x_+^λ and x_-^λ are defined for $\lambda < -1, \lambda \neq -2, -3, \ldots$ with $(x_+^\lambda)' = \lambda x_+^{\lambda-1}$ and $(x_-^\lambda)' = -\lambda x_-^{\lambda-1}$ and distribution $|x|^\lambda$ is defined with $|x|^\lambda = x_+^\lambda + x_-^\lambda$.

Neutrix composition of the distributions x^λ and x_+^μ za $\lambda = -1, -2, \ldots, \mu > 0$ and $\lambda\mu \in \mathbb{Z}^-$

The distributions x_+^r and x_-^r are defined by:

$$x_+^r = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}, \quad x_-^{-r} = -\frac{(\ln x_-)^{(r)}}{(r-1)!} \tag{5}$$

for $r = 1, 2, \ldots,$

Neutrix composition of the distributions x^λ and x_+^μ for $\lambda = -1, -2, \ldots, \mu > 0$ and $\lambda\mu \in \mathbb{Z}^-$

Lemma: Let $\rho(x)$ is infinitely differentiable function. For $s \in \mathbb{Z}^+$ we have:

$$\int_{-1}^{1} v^i \rho^{(s)}(v)dv = \begin{cases} 0, & 0 \le i < s \\ (-1)^s s!, & i = s \end{cases}, \tag{6}$$

$$\int_{-1}^{0} v^s \rho^{(s)}(v)dv = \int_{0}^{1} v^s \rho^{(s)}(v)dv = \frac{1}{2}(-1)^s s! \tag{7}$$

$$\int_{0}^{1} v^s \ln|v| \rho^{(s)}(v)dv = \frac{1}{2}(-1)^s s!\phi(s) + (-1)^s s! c(\rho) \tag{8}$$

$$\int^{1} v^s \ln(1-v)dv = -\frac{\phi(s)}{s}. \tag{9}$$

Neutrix composition of the distributions x^λ and x_+^μ for $\lambda = -1, -2, \dots, \mu > 0$ i $\lambda\mu \in \mathbb{Z}^-$

In the previous lemma the integrals are calculated for $s = 0, 1, 2, \dots,$

where $c(\rho) = \int\limits_0^1 \ln t \rho(t) dt, \phi(s) = \begin{cases} \sum\limits_{k=1}^{s} \frac{1}{k} & s \geq 1 \\ 0 & s = 0 \end{cases}.$

Neutrix composition of the distributions x^λ and x_+^μ for $\lambda = -1, -2, \ldots, \mu > 0$ and $\lambda\mu \in \mathbb{Z}^-$

The following theorem is proved:

Theorem

The distribution $(x_+^\mu)^{-m}$ exists and

$$(x_+^\mu)^{-m} = x_+^{-s} - (-1)^s \frac{(-1)^m m![2c(\rho) + \phi(m-1)] + s\phi(s-1)}{s!}\delta^{(s-1)}(x) \tag{10}$$

for $\mu > 0$, $m = 1, 2, \ldots$ and $\mu m = s (s \in \mathbb{Z}^+)$.

Neutrix composition of the distributions x^λ and x_+^μ for $\lambda = -1, -2, \dots, \mu > 0$ and $\lambda\mu \in \mathbb{Z}^-$

Corollary: The distribution $(x_-^\mu)^{-m}$ exists and

$$(x_-^\mu)^{-m} = x_-^{-s} + \frac{(-1)^m m![2c(\rho) + \phi(m-1)] + s\phi(s-1)}{s!}\delta^{(s-1)}(x), \tag{11}$$

for $\mu > 0, m = 1, 2, \dots$ and $\mu m = s \in \mathbb{Z}^+$.

Corollary: Let by $F_r(x)$ we denote the distribution x_-^{-r}, then the distribution $F_r(x_+^{1/r})$ exists and

$$\left(x_+^{\frac{1}{r}}\right)^{-r} = x_+^{-1} - (-1)^r r![2c(\rho) + \phi(r-1)]\delta(x) \tag{12}$$

for $r = 1, 2, \dots,$ where $\phi(r)$ and $c(\rho)$ are defined in the previous lemma.

Neutrix composition of the distributions x_+^μ, $|x|^\mu$, x^{-s} and $x^{-s}\ln|x|$

We have proved that:

Theorem

The composition of the distributions $(x_+^\mu)_-^{-s}$ exists and

$$(x_+^\mu)_-^{-s} = \frac{(-1)^{m+s}c(\rho)}{\mu(m-1)!}\delta^{(m-1)}(x) \tag{13}$$

for $\mu > 0, s = 1, 2, \ldots,$ where $\mu s = m \in \mathbb{Z}^+$.
Special case:

$$\left(x_+^{\frac{1}{s}}\right)_-^{-s} = (-1)^{s+1}sc(\rho)\delta(x).$$

Neutrix composition of the distributions x_+^μ, $|x|^\mu$, x^{-s} and $x^{-s}\ln|x|$

The result is given in:

Theorem

The composition of the distributions $(|x|^\mu)_-^{-s}$ exists and

$$(|x|^\mu)_-^{-s} = \frac{2(-1)^{m+s}c(\rho)}{\mu(m-1)!}\delta^{(m-1)}(x) \tag{14}$$

for $\mu > 0, s = 1, 2, \dots,$ where $\mu s = m = 1, 3, 5, \dots,$ and

$$(|x|^\mu)_-^{-s} = 0 \tag{15}$$

for $\mu > 0, s = 1, 2, \dots,$ where $\mu s = m \neq 1, 3, 5, \dots.$ Special case $(|x|^{\frac{1}{s}})_-^{-s} = 2sc(\rho)\delta(x).$

Neutrix composition of the distributions x_+^μ, $|x|^\mu$, x^{-s} and $x^{-s}\ln|x|$

By using of the theorem which refers to the compositions $(x_+^\mu)^{-s}$, the corrolaries follow:

Corollary: The compositions of the distributions $(x_+^\mu)_+^{-s}$ exists and

$$(x_+^\mu)_+^{-s} = x_+^{-m} - (-1)^m [L_{m,s}^* + \frac{c(\rho)}{\mu(m-1)!}]\delta^{(m-1)}(x) \qquad (16)$$

for $\mu > 0, s = 1, 2, \dots$ i $\mu s = m \in \mathbb{Z}^+$., where $L_{m,s}^* = \frac{(-1)^s s![2c(\rho)+\phi(s-1)]+m\phi(m-1)}{m!}$.

Neutrix composition of the distributions x_+^μ, $|x|^\mu$, x^{-s} and $x^{-s}\ln|x|$

Corollary: The composition of the distributions $(|x|^\mu)_+^{-s}$ exists and

$$(|x|^\mu)_+^{-s} = |x|^{-m} + [L_{m,s}^* - \frac{2(-1)^m c(\rho)}{\mu(m-1)!}]\delta^{(m-1)}(x), \qquad (17)$$

for $\mu > 0, s = 1, 2, \ldots$ and $\mu s = m = 1, 3, 5, \ldots,$ i

$$(|x|^\mu)_+^{-s} = |x|^{-m} + L_{m,s}^* \delta^{(m-1)}(x) \qquad (18)$$

for $\mu > 0, s = 1, 2, \ldots$ and $\mu s = m \neq 1, 3, 5, \ldots$.

Neutrix composition of the distributions x_+^μ, $|x|^\mu$, x^{-s} and $x^{-s}\ln|x|$

By Fisher and Nicholas the following theorem is proved:

Theorem

The composition of the distributions $(x_+^r)^{-s}$ exists and

$$(x_+^r)^{-s} = x_+^{-rs} + K_{r,s}\delta^{(rs-1)}(x) \qquad (19)$$

for $r, s = 1, 2, \ldots,$ where
$K_{r,s} = (-1)^{rs-1}\frac{(-1)^s s![2c(\rho)+\phi(s-1)]+rs\phi(rs-1)}{(rs)!}$.

Lemma: If φ is arbitrary function in $\mathcal{D}[-1,1]$. Then:

$$\langle x_+^{-s}, \varphi(x)\rangle = \int_0^1 x^{-s}[\varphi(x) - \sum_{i=0}^{s-1} \frac{\varphi^{(i)}(0)}{i!} x^i] dx -$$

$$- \sum_{i=0}^{s-2} \frac{\varphi^{(i)}(0)}{i!(s-i-1)} - \frac{\phi(s-1)\varphi^{(s-1)}(0)}{(s-1)!}, s = 1, 2, \ldots, \quad (20)$$

$$\langle x_+^{-s} \ln x_+, \varphi(x)\rangle = \int_0^1 x^{-s} \ln x[\varphi(x) - \sum_{i=0}^{s-1} \frac{\varphi^{(i)}(0)}{i!} x^i] dx -$$

$$- \sum_{i=0}^{s-2} \frac{\varphi^{(i)}(0)}{i!(s-i-1)^2} - \frac{\phi_1(s-2)\varphi^{(s-1)}(0)}{(s-1)!}, s \geq 2, \phi_1(s) = \sum_{i=1}^{s+1} \frac{\phi(i)}{i}. \quad (21)$$

Neutrix compositions of the distributions x_+^μ, $|x|^\mu$, x^{-s} and $x^{-s}\ln|x|$

We have proved the following theorem:

Theorem

The composition of the distrinutions $x^{-s}\ln|x|$ and x_+^r exists and

$$(x_+^r)^{-s}\ln|x_+^r| = rx_+^{-rs}\ln x_+ + K_{r,s}^*\delta^{(rs-1)}(x), \tag{22}$$

for $s = 1, 2, \ldots$ where $c_1(\rho) = \int\limits_0^1 \ln^2 t\rho^{(s-1)}(t)dt$

$$K_{r,s}^* = \frac{(-1)^{rs-1}}{(rs-1)!}\{\frac{[1+(-1)^{s+1}]c_1(\rho)}{2(s-1)!} + \phi(s-1)[K_{r,s} + \phi(rs-1)]\}.$$

THANK YOU FOR YOUR ATTENTION!