Solutions and comments on **101.I, 101.J, 101.K, 101.L** (November 2017).

101.I (Michel Bataille)

Let a, b, c be the side-lengths of a triangle ABC, r its inradius and R its circumradius. Prove that

$$\sqrt{\frac{2r}{R}} \leqslant \frac{\sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)}}{a+b+c} \leqslant 1.$$

This refinement of Euler's inequality $R \geqslant 2r$ was clearly enjoyed by solvers. The right-hand inequality is quick to establish: the AM-GM inequality shows that

$$\sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)}$$

$$\leqslant \tfrac{1}{2}(b+c+c+a+a+b)$$

$$= a+b+c,$$

as required.

Solvers used a variety of methods for the harder left-hand inequality. The shortest proofs were trigonometric ones along the following lines. From the half-angle formula $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{a(s-a)}{4Rr}}$, since $abc = 4Rrs$. The left-hand inequality is thus equivalent to

$$\frac{\sqrt{2Rr}}{s}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \geqslant \sqrt{\frac{2r}{R}}$$

or $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geqslant \frac{s}{R}$.

This result can be found in the literature. Martin Lukarevski proved it as follows:

By the AM-GM inequality

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geqslant 3\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 3\sqrt[3]{\frac{s}{4R}},$$

since $s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

It then suffices to show that $3\sqrt[3]{\frac{s}{4R}} \geqslant \frac{s}{R}$ or $\frac{s}{R} \leqslant \frac{3\sqrt{3}}{2}$; but this is immediate from Jensen's inequality:

$$\frac{s}{R} = \sin A + \sin B + \sin C \leqslant 3\sin\left(\frac{A+B+C}{3}\right) = \frac{3\sqrt{3}}{2}.$$

Correct solutions were received from: S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. T. Q. Hoare, G. Howlett, A. Li, M. Lukarevski (2 solutions), J. A. Mundie, P. Nüesch, V. Schindler, I. D. Sfikas, G. B. Trustrum, L. Wimmer (2 solutions) and the proposer Michel Bataille.