$$\geqslant S + \tfrac{1}{2}xy \sin\theta$$
$$= 2S.$$

There is equality if, and only if, $x = y$ and $\theta = 90°$ so the diagonals of the original quadrilateral are equal and perpendicular just like the Van Aubel line segments.

Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, A. Li, P. Nüesch, I. D. Sfikas, C. Starr, G. Strickland, L. Wimmer and the proposer Isaac Sofair.

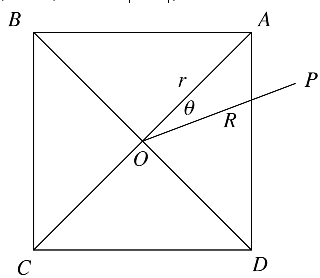
101.K (Abdurrahim Yilmaz)

Consider a square $ABCD$ with centre O, circumradius r and a point P in the plane of the square with $|OP| = R$. Let $\alpha, \beta, \gamma, \delta$ denote the areas of the triangles with side-lengths $|PA|, |PB|, |PC|$; $|PA|, |PB|, |PD|$; $|PA|, |PC|, |PD|$ and $|PB|, |PC|, |PD|$ respectively, whenever these triangles exist. Prove that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \tfrac{1}{4}\left[3\left(R^2 + r^2\right)^2 - 10R^2r^2\right].$$

All respondents deftly handled the potentially heavy algebra in this problem and most solutions went along the following lines. With the notation as in the figure and writing $a = |PA|, \ldots, d = |PD|$, we have

$$a^2 = R^2 + r^2 - 2Rr\cos\theta$$
$$b^2 = R^2 + r^2 + 2Rr\sin\theta$$
$$c^2 = R^2 + r^2 + 2Rr\cos\theta$$
$$d^2 = R^2 + r^2 - 2Rr\sin\theta$$



from which

$$\sum a^4 = 4\left(R^2 + r^2\right)^2 + 8R^2r^2$$

and
$$\sum a^2b^2 = \frac{1}{2}\left[\left(\sum a^2\right)^2 - \sum a^4\right]$$
$$= \frac{1}{2}\left[16\left(R^2 + r^2\right)^2 - 4\left(R^2 + r^2\right)^2 - 8R^2r^2\right]$$
$$= 6\left(R^2 + r^2\right)^2 - 4R^2r^2.$$

Heron's formula gives

$$\alpha^2 = s(s-a)(s-b)(s-c)$$
$$= \tfrac{1}{16}\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right)$$

and so

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \tfrac{1}{16}\left(4\sum a^2b^2 - 3\sum a^4\right)$$
$$= \tfrac{1}{16}\left[24\left(R^2 + r^2\right)^2 - 16R^2r^2 - 12\left(R^2 + r^2\right)^2 - 24R^2r^2\right]$$
$$= \tfrac{1}{4}\left[3\left(R^2 + r^2\right)^2 - 10R^2r^2\right].$$

James Mundie provided a graph showing the region where one or more of the triangles with sides $|PA|$, $|PB|$, $|PC|$, etc. does not exist which he described as a 'cuspy annulus' containing the points A, B, C, D. And the proposer, Abdurrahim Yilmaz noted that $\dfrac{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}{r^4}$ attains a minimum value of $\tfrac{5}{12}$ when $R = \sqrt{\tfrac{2}{3}}r$.

Correct solutions were received from: M. Bataille S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, M. Lukarevski, J. A. Mundie, G. Strickland, L. Wimmer and the proposer Abdurrahim Yilmaz.

101.L (Finbarr Holland)

Suppose G and K denote the centroid and the Lemoine point of a triangle ABC. Prove that, unless ABC is equilateral, at least one of the ratios

$$\frac{|AG|}{|AK|}, \frac{|BG|}{|BK|}, \frac{|CG|}{|CK|}$$

exceeds 1.

Vectors, complex numbers and trilinear/areal coordinates all featured in the solutions to this problem.

If $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{g}$, $\mathbf{k}$ are the position vectors of A, B, C, G, K and if $a = |BC|$, etc. with the usual triangle notation, then, by standard results, we have

$$3\mathbf{g} = \mathbf{p} + \mathbf{q} + \mathbf{r} \quad \text{and} \quad \left(a^2 + b^2 + c^2\right)\mathbf{k} = a^2\mathbf{p} + b^2\mathbf{q} + c^2\mathbf{r}.$$

Then

$$9\left|\overrightarrow{AG}\right|^2 = (\mathbf{p} + \mathbf{q} + \mathbf{r} - 3\mathbf{p})^2$$
$$= (\mathbf{q} - \mathbf{p} + \mathbf{r} - \mathbf{p})^2$$
$$= c^2 + b^2 + 2(\mathbf{q} - \mathbf{p})\cdot(\mathbf{r} - \mathbf{p})$$
$$= c^2 + b^2 + b^2 + c^2 - a^2, \text{ by the cosine rule}$$
$$= 2b^2 + 2c^2 - a^2.$$