# Simple OLG Model Explained With Numerical Example in MATLAB

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**Abstract:** In this paper it is been made an attempt to map the global OLG models and afterwards to draw phase diagrams for the baseline OLG models, namely by simulating the Diamond (1965) capital accumulation simple OLGmodel. First it is discussed the issue of optimality in OLG models, then the general framework of the models is being mapped, with the issue of overaccumulation in OLG models. In the next section this paper presents dynamic general equilibrium analysis of an overlapping generations models in which each individual lives in two periods lifecycle. This represents the simplest of OLG models. An overlapping generations model is an applied DGE model for which the lifecycle models are applied. In the applied part benchmark models has been compared to, model with parameters that generate poverty traps and multiple equilibria.

**Keywords:** OLG models, DGE models, MATLAB codes for OLG, simplest OLG model, poverty traps and multiple equilibria OLG models

## Introduction and literature review

The basic OLG model with capital accumulation is due to Allais (1947), and Diamond (1965).Samuelson (1958) introduced a consumption loan model to analyze the interest rate ,with or without social contrivance of money has developed into one of the most significant paradigm of the neoclassical general equilibrium theory ,by passed Arrow-Debreu(1954) economy, Geanakopolos, (1987) . The concept of OLG models has been inspired by the Irving Fisher's Theory of interest (1930).OLG models belong to the class of intertemporal general equilibrium models, the OLG model has become strongest competitor of the Arrow-Debreu paradigm. However, OLG models retain the most important neoclassical assumptions. Namely agents maximize objective functions utility or profit functions subject to budget or technology constraints, agents are price takers, agents have perfect foresight i.e. there are rational expectations in the presence of uncertainty, there is also market clearing situation in the model. The basic OLG model of an exchange economy is due to Samuelson (1958),de la Croix, D.Michel, P. (2002) .One particular central feature of the OLG models is that steady-state equilibrium need not be Pareto efficient[1]. Though not every equilibrium is inefficient, the efficiency of the equilibrium depends on the Cass-criterion[2], see Cass,(1972).This criterion gives necessary and sufficient condition when OLG competitive equilibrium allocation is inefficient. Another problem in OLG models is the one proposed by Diamond (1965) and it's about over saving[3] which occurs when capital accumulation is added to the model. In the terminology of Phelps (1961), the capital stock exceeds the Golden rule level[4].Weil (1987) , also argues that dynamic efficiency is necessary condition for the RET theorem of Barro (1974) to hold.Pareto optimal solution when $k^* > k^G$(dynamically inefficient economy), can be obtained if the current generation is allowed fast consumption (capital devouring), while future generation to hold their consumption

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[1] First fundamental theorem works with either finite number of agents, or finite number of time periods. This is the

theorem saying that when increasing returns to scale are absent, markets are competitive and complete, no goods are of public good character, and there are no other kinds of externalities, then market equilibria are Pareto optimal. In fact, however, the First Welfare Theorem also presupposes a finite number of periods or, if the number of periods is infinite, then a finite number of agents.

[2] Feasible path $k_t$ is inefficient if and only if $\lim_{t\to\infty} \sum_{t=0}^{t} p_t < \infty$

[3] Over saving occurs when $s^* > \frac{\left(\frac{df(k)}{dk}\right)}{\frac{f(k)}{k}}$, where $s^*$ represents the golden rule saving

[4] $\frac{dk}{dt} > sf(k) - nk$ or $\frac{dk}{dt} > f(k) - c - nk$ ,or $f(k) > n + p$ see Appendix 1 for derivation of the results for the Golden Utility growth compared to Golden rule growth and Ramsey exercise

constant,Mankiw;N.G. Summers,L. Zeckhauser R.J.(1989) .OLG models also enables us to look at the intergenerational redistribution such as Social security but that is beyond the scope of this paper. Another update that was made available for this model was the perpetual youth assumption. Probability of death of an agent is constant, independent of agents age, Ascari,G. Rankin, N.(2004).This model was built by Blanchard (1985), on the foundations laid in Yaari (1965), and in this framework the Ricardian equivalence hypothesis does not work, and this represents nice framework for analyzing government debt and deficits[5]. These models are also beyond the scope of this paper as we will stick in the simulation model to Diamond (1965) work.

## The issue of optimality and the OLG

In the economy without public goods and externalities the competitive equilibrium is Pareto optimal (First fundamental welfare theorem), Arrow (1951) , Debreu (1954). This previous property is not necessary verified when there are infinite agents and infinite number of periods when they exist. About the notion of efficiency one can write in the OLG context :

### Equation 1

$$c_t^y(s), c_t^o(s)' \to PE \Leftrightarrow \nexists\{c^y(s^\infty), c^o(s)'\}_{t=0}^{\infty}$$

So that :

### Equation 2

$U(c^y, c^o, s_t) \leq U(c_t^y, c_{t+1}^o, s^t), \forall t, \vee s^\infty \in S^\infty$i.e. $\forall s^\infty \in \tilde{A}, \exists t, \vee U(c^y, c^o, s_t) < U(c_t^y, c_{t+1}^o, s^t)$

Where $A \in S$ ,$P(s, ds')$ represents the stochastic kernel which describes the system evolution from one S to another, where s represents state-space, Barbie, Kaul (2015). In the formal statement of the theorem it is said: If preferences are locally nonsatiated ($\gtrsim on\ X$[6]is locally nonsatiated if $\forall x \in X, \epsilon > 0, \exists y \in X, \|y - x\| < \epsilon, \wedge u(y) > u(x)$) , and if $x^*, y^*, p$ is a price equilibrium with transfers ,then the allocation $(x^*, y^*)$ is Pareto optimal. The vector $x = (x_1, x_2, \ldots, x_n)$ is a price equilibrium,Mas-Colell, A. (1986).:

### Equation 3

$$\Leftrightarrow \exists p \in R^n, R_{++}^n = \{x \in R^n : x \gg 0\} p \neq 0, p > 0, \lim_{t \to \infty} \sum_{t=0}^{t} p_t < \infty, v \succ_i x_i \to p \cdot v > p \cdot x_i, \forall i$$

The indirect utility function is defined as:$for\ p, \omega \gg 0, v(p, \omega) = u(\varphi(p, \omega))$

The inverse aggregate demand function $\varphi(p, \omega)$ , satisfies the following properties:

a) $\varphi(\cdot)$ is continuous function on $\mathbb{R}_{++}^n \times \mathbb{R}_+^n$

b) $\varphi(\cdot)$ is homogenous of degree zero , $\varphi(\alpha p, \alpha\omega) = \varphi(p, \omega), \forall(p, \omega), * > 0$

c) $\bar{u} \in u(R_{++}^l)$ is homogenous of degree one, concave, and of class , $C^1$ (continuous differentiable whose derivative is continuous ,i.e. continuously differentiable, $\partial e_{\bar{u}}(p) = hu(p)$.

The indirect utility function is the inverse of the expenditure function $v(e_{\bar{u}}(p)) \equiv u$,Varian (1992).Utility in the social welfare function provides a guideline for the government for achieving optimal distribution of income, Tresch, R. W. (2008).Social welfare functions can be defined as:

a) $SWF = \int U^i\, di$-Utilitarian or Benthamite

b) $SWF = min_i U^i$- Rawlsian

5 This framework has been used long-run in the study of economic growth, and short-run context n the business cycle theory.

6 Preference relation $\gtrsim$ is a relation $\gtrsim \subset \mathbb{R}_+^l \times \mathbb{R}_+^l$.With properties $x \gtrsim x, \forall x \in \mathbb{R}_+^l$ (reflexivity), $x \gtrsim y, y \gtrsim z \Rightarrow x \gtrsim z$ (transitivity), $\gtrsim$ is a closed set (continuity), $\forall(x \gtrsim y), \exists(y \gtrsim x)$ (completeness) ,given $\gtrsim, \forall\ (x \gg 0)$the at least good set $\{y: y \gtrsim x\}$is closed relative to $R^l$ (boundary condition), $A$ is convex, $if$ $\{y: y \gtrsim x\}$is convex set for every y , $ay + (1 - \lambda)x \gtrsim x$, whenevery $\gtrsim$ x $and\ 0 < a < 1$,Mas-Colell, A. (1986).

c) $SWF = \int U^i di \rightarrow G(U) = \frac{U^{1-\gamma}}{1-\gamma}$ if $\gamma = 0$ function is utilitarian ,Rawlsian if $\gamma = \infty$

d) With Pareto weights: $SWF = \int \mu_i U^i \, di$ where $\mu_i$ is exogenous.

Measure for assessment of the allocative efficiency when trade takes place is the Conditional Pareto optimality (CPO),Chattopadhyay,S. Gottardi,P.(1999).This notion was proposed by Muench, T. J.(1977). In his paper Muench, T. J.(1977) proves that Lucas equilibrium, Lucas, R. Jr.,(1972) is not Pareto optimal. Muench, T. J.(1977) , uses much stronger Pareto optimality criterion than Lucas ,(1972),even by Lucas, (1977) own words. This criterion is known as Equal treatment Pareto optimal criterion, ET-PO: $\int u\big(c\big((\vartheta)\,,\ell(\vartheta)\big) + v[c^1(\vartheta,\vartheta^1)]\vartheta\phi(\vartheta)\phi(\vartheta^1)d\vartheta d\vartheta^1$.

Where $\ell$ represents the labor supply, and $\ell - c = \varphi$, $\varphi$ is output which is put to the market, $c^1$ represents the old age consumption , $c$ alone represents the young age consumption, $\phi$ represents probability density function[7], $(\vartheta, \vartheta^1)$ represent the allocative distributions of agents when young $\vartheta$ and old $\vartheta^1$ respectively. Younger cohort is divided in two groups following: $\left(\frac{\vartheta}{2}\right)\ell + \left(\frac{2-\vartheta}{2}\right)\ell = \ell, 0 \leq \vartheta + \leq 2$.And $u(c,\ell) = w(c, L - \ell)$ ,in previous expression $(L-\ell)$ is used to denote leisure $L$ is labor used in the production process. An ET-PO condition requires: $\varphi(\vartheta) = \ell(\vartheta^1) - c(\vartheta) = \frac{\vartheta c^1(\vartheta,\vartheta^1)+(2-\vartheta)c^1(2-\vartheta,\vartheta^1)}{2\vartheta^1}$.ET-PO allocation maximizes $\int u\big(c(\vartheta)\,,\ell(\vartheta)\big) + v[c^1(\vartheta,\vartheta^1)]\vartheta\phi(\vartheta)\phi(\vartheta^1)d\vartheta d\vartheta^1$[8] s.t. $\frac{\vartheta c^1(\vartheta,\vartheta^1)+(2-\vartheta)c^1(2-\vartheta,\vartheta^1)}{2\vartheta^1}$. Corresponding Lagrangian is given as:

**Equation 4**

$$\mathcal{L} = \int u\left(c(\vartheta'),\ell(\vartheta')\right)\vartheta' + \lambda\left(\vartheta'\right)\left[\ell\left(\vartheta'\right) - c\left(\vartheta'\right)\right]\phi\left(\vartheta'\right)d\left(\vartheta'\right) + \int v\left[c'\left(\vartheta,\vartheta'\right)\right]\vartheta - \lambda\left(\vartheta,\vartheta'\right)\frac{\vartheta c'\left(\vartheta,\vartheta'\right)+(2-\vartheta)c'\left(2-\vartheta,\vartheta'\right)}{2\vartheta'}\phi(\vartheta)\phi\left(\vartheta'\right)d\vartheta d\vartheta'$$ or

**Equation 5**

$$\nabla_{c,\ell,v}\mathcal{L}(c,\ell,v) = \left(\frac{\partial\mathcal{L}}{\partial u},\frac{\partial\mathcal{L}}{\partial\ell},\frac{\partial\mathcal{L}}{\partial v}\right) = \left(\frac{\lambda(\vartheta')}{\vartheta'},-\frac{\lambda(\vartheta')}{\vartheta'},\frac{\lambda(\vartheta')}{\vartheta'}\right)$$

Unique solution is $\frac{\lambda(\vartheta')}{\vartheta'}$. The support of the pdf is $\phi(\vartheta) \in (\epsilon, 2-\epsilon), 0 < \epsilon < 1$ .In the Lucas ,(1972)allocation case : $\int \frac{\gamma(\vartheta)}{\vartheta}\frac{v[c^1(\vartheta,\vartheta^1)]}{\rho(\varphi(\vartheta^1))}\phi(\vartheta)d\vartheta = \frac{\gamma(\vartheta')}{\vartheta'}$. In the previous expression $\rho\big(\varphi(\vartheta^1)\big) = v'[\vartheta'\varphi(\vartheta')]$, also from previous expression: $\frac{\gamma(\vartheta)}{\vartheta}v\left[c^1\left(\vartheta,\vartheta^1\right)\right] = \frac{\gamma(2-\vartheta)}{2-\vartheta}\,v\left[c^1\left(2-\vartheta,\vartheta^1\right)\right]$

Previous two conditions (Eq.11, Eq12) are a must for L-allocation. The F-function is introduced : $F(\vartheta,\vartheta') = \frac{\left(\left(\frac{\vartheta'}{\vartheta}\right)\varphi(\vartheta')\right)v'\left(\left(\frac{\vartheta'}{\vartheta}\right)\varphi\right)\phi(\vartheta')}{\varphi(\vartheta)v'\left[\vartheta'\varphi\left(\vartheta'\right)\right]}$ By the Lucas ,(1972) : $\int\frac{\left(\left(\frac{\vartheta'}{\vartheta}\right)\varphi(\vartheta')\right)v'\left(\left(\frac{\vartheta'}{\vartheta}\right)\varphi\right)\phi(\vartheta')}{\varphi(\vartheta)v'\left[\vartheta'\varphi\left(\vartheta'\right)\right]}d\vartheta' = 1$ . So the previous conditions (Eq.11, Eq12) will become : $\int\tilde{\gamma}(\vartheta)F(\vartheta,\vartheta')d\vartheta = \tilde{\gamma}(\vartheta)$ and $\int\tilde{\gamma}(\vartheta)F(\vartheta,\vartheta')d\vartheta = \int\tilde{\gamma}\,(2-\vartheta)F(2-\vartheta,\vartheta')d\vartheta'$ . This expression $\int\tilde{\gamma}\,(2-\vartheta)F(2-\vartheta,\vartheta')d\vartheta$ when integrated becomes : $\int\tilde{\gamma}\,(2-\vartheta)F(2-\vartheta,\vartheta')d\vartheta = \tilde{\gamma}(2-\vartheta)$. Than previous expression must satisfy, from previous one remembers that $0 < \epsilon < 1$: $\int_{\varepsilon\leq\vartheta}^{\varepsilon\leq\vartheta\leq1}\tilde{\gamma}(\vartheta)[\,F(\vartheta,\vartheta') - F(\vartheta,2-\vartheta')]d\vartheta = 0$ From the previous expression:

**Equation 6**

$$\frac{\left(\left(\frac{\vartheta'}{\vartheta}\right)\varphi(\vartheta')\right)v'\left(\left(\frac{\vartheta'}{\vartheta}\right)\varphi(\vartheta')\right)}{\varphi(\vartheta)v'[\vartheta'\varphi(\vartheta')]} - \left(\frac{(2-\vartheta)}{\vartheta}\right)\varphi(2-\vartheta')v'\left[\frac{(2-\vartheta')}{\vartheta}\varphi(2-\vartheta')\right]$$

[7] First derivative of CDF

[8] Integral over $\phi(\vartheta)\phi(\vartheta^1)$ domain is equal to one, since this is a PDF .

Since $\frac{d[\vartheta'\varphi(\vartheta')]}{d\vartheta'} > 0$, so the expression in brackets in previous expression is positive, strictly positive. So this expression to be true $\int_{\varepsilon\leq\vartheta}^{\varepsilon\leq\vartheta\leq 1} \tilde{\gamma}(\vartheta)[\, F(\vartheta,\vartheta') - F(\vartheta, 2-\vartheta')]d\vartheta = 0$ , must be that $\tilde{\gamma}(\vartheta) \equiv 0$, which also implies that $\gamma(\vartheta) \equiv 0$ which is contradictory. Time $t$ in the model is discrete and runs from $1 \to \infty$.In each period there exist realized state $\exists s_t$. The beginning state one can consider to be given $s_0 \in S$, Ohtaki,E.,(2013).Agents that enter in the model newborns are element of the nonempty finite set of agents $h \in H$. Endowment of agents born in two states (two time periods) is given by:

### Equation 7

$$\omega_h^s = \left(\omega_s^{h_1}, \left(\omega_{ss^1}^{h_2}\right) s^1 \in S\,\right) \in \mathbb{R} \times \mathbb{R}_+^s, u = \mathbb{R}_+ \times \mathbb{R}_+^s \to \mathbb{R}$$

Where utility function is quasi-concave strictly quasi concave (the negative of quasiconvex) , monotone utility functions. $f: \mathbb{R}^l \to R$ is strictly quasi concave if $f(\lambda x_1 + (1-\lambda)x_2) > \min\{f(x_1), f(x_2)\}$,holds for all $x_1$ , $x_2 \in \mathbb{R}^l$,[9] with $x_1 \neq x_2$ and all $\lambda \in (0,1)$,Quasi concave function would be $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$,Josheski, D. (2017). Utility function also I twice differentiable[10].Total or maximal endowment is given by expression :$\bar{\omega}_{ss'} := \sum_{h\in H}\left(\omega_s^{h_1} + \omega_{ss^1}^{h_2}\right)$.

Stationary feasible allocations are given as :$\sum_{h\in H}\left(c_s^{h_1} + c_{os^1}^{h_2}\right) = \bar{\omega}_{ss'} c_s^{h_1}; c_{os^1}^{h_2}: S \to \mathbb{R}_+, \left(c_s^h \left(c_{ss^1}^{h_2}\right)_{s'\in s}\right) \in \mathbb{R} \times \mathbb{R}_+^s$.Now let A be the convex set of all stationary and feasible allocations : $A$ is convex, $if$ $\{\mathrm{y}: \mathrm{y} \gtrsim \mathrm{x}\,\}$is convex set for every y , $ay + (1-\lambda)x \gtrsim x$, whenevery $\gtrsim$ x $and$ $0 < a < 1$. And :

*Lemma 1*:

### Equation 8

$$c_s^{h_1}; c_{os^1}^{h_2} \subset \mathbb{R}_+, c \in \left\{c_s^{h_1}; c_{os^1}^{h_2}\right\}, \exists r > 0, B_r(c) = \left\{c_s^{h_1} \middle| \left|c_s^{h_1} - c_{os^1}^{h_2}\right| \leq r\right\} \subset c$$

Where $r = radius$ of a ball $B_r$and $intc \subset c \subset clc$ , where $clc$ is a closure of $c$. From previous now we define theorem 1 which states that $CPO^* \subset CGRO^*$ that conditional Pareto optimal and conditional golden rule allocations respectively. And now Proposition 1:

a) $X = (x^0, x^1), Y = (y^0, y^1) \in R_+^s \times R_+$,are stationary feasible allocations ,$x^1 \neq y^1$and in $ay + (1-\lambda)x \gtrsim x$ ,$\alpha \in [0,1]$,$U^{hs}$: $ay + (1-\lambda)x > \{U^{hs}(x), U^{hs}(y)\}$,or $x^0 \neq y^0$ and in $\forall(h,s) \in H \times S, ay + (1-\lambda)x \gtrsim x$ ,$\alpha \in [0,1]$,$U^{hs}$: $ay + (1-\lambda)x > \{U^{hs}(x), U^{hs}(y)\}$

Proof of the following proposition is given as;

b) $\forall(h,s) \in H \times S\,, \exists b \in A, b_{0s}^{h_0} \geq c_{0s}^{h_0}, U^{hs}\left(b_s^h\right) \geq U^{hs}\left(c_s^h\right)$, if $b$ satisfies that $\exists(h,s) \in H \times S$,then $c$ is not CGRO since its not CPO. And then about stationary feasible allocations we assume that : $\forall s \in s', \bar{\omega}_{s_o s'} - \sum_{h\in H} b_{os'}^{h_1} = \sum_{h\in H} b_{s'}^{h_0} = \sum_{h\in H} c_{s'}^{h_0} = \bar{\omega}_{s_o s'} - \sum_{h\in H} c_{ss'}^{h_1} \Leftrightarrow \sum_{h\in H} b_{os'}^{h_1} = c_{ss'}^{h_1} \neg\, b_{0s}^{h_0} \geq c_{0s}^{h_0}, \forall(h,s\,) \equiv \sum_{h\in H} b_{os'}^{h_1} \geq c_{ss'}^{h_1}$also from $U^{hs}$: $ay + (1-\lambda)x > \{U^{hs}(x), U^{hs}(y)\}$ quasi-concave utility functions and if we let $d := ac + (1-\lambda)b$ , $a \in [0,1]$, it follows that $\sum_{h\in H} d_{s'}^{h_0} + \sum_{h\in H} d_{ss'}^{h_1} = \alpha \sum_{h\in H}\left(c_{s'}^{h_0} + c_{ss'}^{h_1}\right) + (1-\alpha)\sum_{h\in H}\left(b_{s'}^{h_0} + b_{ss'}^{h_1}\right) = \bar{\omega}_{ss'}$

The states in the previous expressions are following Markov process with time invariant probabilities,Aiyagari and Peled (1991), like these : $\pi^{ss'} = Prob\{s_1 = s' | s_0 = s\}, (s,s') \in S$ .And some allocation $c_{\max}$ $\in C$ Pareto dominates $c \in C, \forall h \in H$ :$\bar{c}_{1o}^h(s) \geq \bar{c}_1^h(s), s \in S$ and that that translates to:

### Equation 9

$$\int_s \int_{s'} \pi^{ss'}\, u^h[\bar{c}_{1o}^h(s), \bar{c}_1^h(s), s, s']dsds' \geq \int_s \int_{s'} \pi^{ss'}\, u^h[c_{1o}^h(s), c_1^h(s), s, s']dsds', s \in S$$

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9 $l = s$

10 $\frac{\partial u^{hs}(c^y,c^o)}{\partial c^y}, \frac{\partial u^{hs}(c^y,c^o)}{\partial c_{s'}^o}, \forall h \in H, c^y, c^o \in R_+ \times R_+^s$

Previous expression is strict inequality somewhere in the interior allocation. Space allocations on the other hand are $:X = \{x \in \Pi_{i\in\vartheta} x^h : \sum_{i\in\vartheta} x^h \in \ell\}$.Markets are well defined Arrow-Debreu prices or contingent claim prices, $P = \{p \in \mathbb{R}^s : p_h > 0, \forall h \in H \subset S\}$.And allocation $x \in X$ is robustly inefficient if it is Pareto dominated by an alternativeallocation $\vartheta \in X$ ,$0 < \epsilon < 1$ , Bloise, G., Calciano, F.L.(2008):

**Equation 10**

$$\sum_{h\in H\subset\vartheta} (\vartheta^h - x^h) \leq (1-\epsilon) \sum_{h\in H\subset\vartheta} (\vartheta^h - x^h)$$

And $\exists T_p(\ell)_h = \frac{1}{p_h}\sum_{h_1\in\vartheta'} p_{h_1}\epsilon_{h_1}$ , and there exists some $\epsilon \in \ell, \epsilon > 0$. Radius of the positive linear operator $T$ from itself to $\ell$ is defined: $r(T) = \lim_{H\in\mathbb{N}} \|T^H\|^{\frac{1}{H}} = \inf_{H\in\mathbb{N}} \|T^H\|^{\frac{1}{H}}$. In the previous expressions $\ell$ is a Banach lattice (a partially ordered Banach space $X$ over time)[11], $\|v\|_1 \leq c\|v\|_2$ and $\|v\|_2 \leq C\|v\|_1$,Banach space is used only in infinite dimensional setting $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$, Renteln, P. and Dundes, A. (2005).

## The model framework

Since both technology and labor supply are growing, one needs to work with intensive form of output:

**Equation 11**

$$y_t = \frac{Y_t}{A_t L_t} = \left(\frac{1}{(1+g)(1+n)}\right)^a k_{t-1}^a$$

Factor markets are competitive and capital and labour are earning their marginal products:

**Equation 12**

$$r_t = a k_{t-1}^{a-1} (A_t L_t)^{1-a} - \delta \;\; W_t = (1-a) K_{t-1}^a (A_t L_t) A_t$$

Because output is generated using a constant return to scale technology there will be zero profits. Next is introduced the intensive form of the zero profit condition.

**Equation 13**

$$y_t = \frac{1}{(1+g)(1+n)} (r_t + \delta) k_t + w_t$$

Utility is a function of consumption when "young" $\frac{c_{1,t}}{L_t}$, and "old" $\frac{c_{2,t+1}}{L_t}$,and function of utility that is maximized is given as:

**Equation 14**

$$\max_{\frac{c_{1,t}}{L_t}, \frac{c_{2,t}}{L_t}} U_t = \frac{\left(\frac{c_{1,t}}{L_t}\right)^{1-\theta}}{1-\theta} + \beta \frac{\left(\frac{c_{2,t}}{L_t}\right)^{1-\theta}}{1-\theta}$$

Subject to following constraints:

**Equation 15**

$$\frac{c_{1,t}}{L_t} + \frac{S_t}{L_t} = W_t; \frac{c_{2,t+1}}{L_t} = (1+r_{t+1})\frac{S_t}{L_t}; c_{1,t} = \frac{c_{1,t}}{A_t L_t}; s_t = \frac{S_t}{A_t L_t}; c_{2,t} = \frac{c_{1,t}}{A_{t+1} L_t}$$

To derive consumption Euler equation one maximizes :

**Equation 16**

$$\max_{c_{1,t}, c_{2,t+1}} \frac{\left(c_{1,t\,A_t}\right)^{1-\theta}}{1-\theta} + \beta \frac{\left(c_{2,t+1\,A_{t+1}}\right)^{1-\theta}}{1-\theta} c_{1,t} + s_t = w_t c_{2,t+1} = \frac{1}{1+g}(1+r_{t+1}) s_t$$

The Lagrangian of the previous optimization problem is given as:

**Equation 17**

$$\mathcal{L} = \max_{s_t} \frac{\left((w_t - s_t)A_t\right)^{1-\theta}}{1-\theta} + \beta \frac{\left(\frac{1}{1+g}(1+r_{t+1}) s_t A_{t+1}\right)^{1-\theta}}{1-\theta}$$

FONCs for $s_t$are given as:

**Equation 18**

$$\frac{\partial\mathcal{L}}{\partial s_t} : -A_t\left((w_t - s_t)A_t\right)^{-\theta} + \beta \frac{1}{1+g}(1+r_{t+1})A_{t+1} + \left(\frac{1}{1+g}(1+r_{t+1}) s_t A_{t+1}\right)^{-\theta}$$

Re-arranging this yields:

**Equations 19**

[11] $x \leq y \Leftrightarrow x + z \leq y + z, \forall x, y, z \in X; \alpha x \geq 0, \forall a \geq 0; \forall x, y, \exists x \vee y, (\mathrm{l.u.b}), \exists x \wedge y, (\mathrm{g.l.b}), \|x\| \leq \|y\|, |x| \leq |y|, |x| = x \vee (-x)$

$$A_t\big((w_t - s_t)A_t\big)^{-\theta} = \beta \frac{1}{1+g}(1 + r_{t+1})A_{t+1}\left(\frac{1}{1+g}(1 + r_{t+1})s_t A_{t+1}\right)^{-\theta}$$

$$(w_t - s_t)^{-\theta} = \beta(1 + r_{t+1})\big((1 + r_{t+1})s_t\big)^{-\theta}$$

$$(w_t - s_t)^{-\theta} = \beta(1 + r_{t+1})^{1-\theta}s_t^{-\theta} w_t - s_t = \beta^{-\frac{1}{\theta}}(1 + r_{t+1})^{\frac{\theta-1}{\theta}} s_t$$

$$s_t(r_{t+1}, w_t) = \frac{w_t}{1 + \beta^{-\frac{1}{\theta}}(1 + r_{t+1})^{\frac{\theta-1}{\theta}}} K_t = (1-\delta)K_{t-1} + s(r_{t+1}, w_t)$$

As in the infinite –horizon model ,the capital stock at the period $t$ is the amount saved by the "young"
individuals in period $t$:

**Equation 20**

$$k_t = \frac{1-\delta}{(1+g)(1+n)} k_{t-1} + s(r_{t+1}, w_t)$$

Or

**Equation 21**

$$\begin{aligned} k_t &= \frac{1}{(1+g)(1+n)}[(1-\delta)k_{t-1} + r_t] \\ &= \frac{1}{(1+g)(1+n)}\left[(1-\delta)k_{t-1} + y_t - c_t^Y - \frac{1}{(1+g)(1+n)}c_t^O\right] \\ &= \frac{1}{(1+g)(1+n)}(1-\delta)k_{t-1} + k_{t-1}^a - c_t^Y - (1 + r_t - \delta)k_{t-1} \\ &= \frac{1}{(1+g)(1+n)}[k_{t-1}^a - c_t^Y - r_t k_{t-1}] \\ &= \frac{1}{(1+g)(1+n)}[(1-\alpha)k_{t-1}^a - c_t^Y] \\ &= \frac{1}{(1+g)(1+n)}[w_t - c_t^Y] \\ &= \frac{s_t}{(1+g)+(1+n)} \end{aligned}$$

Aggregate equations, and the consumption of "young" and "old" is given as follows:

## Equations 22

$C_t^Y = A_{t-1}L_{t-1}c_t^Y$-intensive form
$C_t^O = (1-\delta)K_{t-1} + r_t k_{t-1}$-savings that are spent by "old"
$W_t = \frac{\partial Y_t}{\partial L_{t-1}} = A_{t-1}(1-\alpha)\left(\frac{K_{t-1}}{A_{t-1}L_{t-1}}\right)^{\alpha}$-wage rate
$r_t = \frac{\partial Y_t}{\partial K_{t-1}} = \alpha\left(\frac{K_{t-1}}{A_{t-1}L_{t-1}}\right)^{\alpha-1}$-interest rate
$K_t = (1-\delta)K_{t-1} + I_t$-motion of capital
$I_t = A_{t-1}L_{t-1}r_t$–investment in period $t$
$K_t = A_t L_t k_t$– capital in intensive form
$Y_t = C_t^Y + C_t^O + I_t$-expenditure in period $t$
$Y_t = W_t L_{t-1} + r_t K_{t-1}$-income in period $t$
$S_t = A_{t-1}L_{t-1}s_t$–savings in period $t$ in intensive form
$C_{t+1}^O = \frac{(1+r_{t+1}-\delta)}{(1+g)(1+n)} s_t$-consumption of savings of "old"

The young agent consumption/savings decision problem is:

$$U_t = \frac{(A_{t-1}c_t^Y)^{1-\theta} - 1}{1-\theta} + \beta\frac{(A_{t-1}c_{t+1}^O)^{1-\theta} - 1}{1-\theta}$$

By using $c_t^Y = w_t - s_t$, and the maximization problem as a function of $s_t$can be written as:

**Equation 23**

$$U_t = \frac{(A_{t-1}w_t - A_{t-1}s_t)^{1-\theta} - 1}{1-\theta} + \beta\frac{(A_{t-1}(1 + r_t - \delta)s_t)^{1-\theta} - 1}{1-\theta}$$

By solving we get :

## Equations 24

$$\frac{\partial U_t}{\partial s_t} = 0;\ s_t = \frac{w_t}{1 + \beta^{\frac{1}{\theta}}(1 + r_t - \delta)^{\frac{\theta-1}{\theta}}}$$

Now one can pose the problem fully specified:

$y_t = c_t^Y + \frac{1}{(1+g)(1+n)}c_t^O + r_t$-output and interest rate

$y_t = w_t + r_t k_{t+1}$-output per worker

$k_t = \frac{1}{(1+g)(1+n)}[(1-\delta)k_{t-1} + r_t]$-motion of capital $t$

$r_t = \alpha k_{t-1}^{\alpha-1}$-interest rate at time period $t$

$w_t = (1-\alpha)k_{t-1}^{\alpha-1}$-wage rate at time $t$

$$s_t = \frac{w_t}{1 + \beta^{\frac{1}{\theta}}(1 + r_t - \delta)^{\frac{\theta-1}{\theta}}}$$

$C_{t+1}^O = (1+g)(1+n)(1 + r_t - \delta)k_{t-1}$-consumption of "old"

$c_t^Y + s_t = w_t$–consumption and saving when related with wage rate at time $t$

Output per effective worker and wage are given as:

## Equation 25

$$y = \left(\frac{1}{(1+g)*(1+n)}\right)^{\alpha} * k^{\alpha},\ w = (1-\alpha) * y$$

Interest rate and savings rate are given as:

**Equation 26**

$$r = \alpha * \left(\frac{1}{\left((1 + g) * (1 + n)\right)}\right)^{\alpha - 1} * k^{\alpha - 1} - \delta \,; s = \frac{w}{\left(1 + \beta^{-\frac{1}{\theta}} * (1 + r - \delta)^{\frac{(\theta - 1)}{\theta}}\right)};$$

Consumption of the old cohort is given as and consumption per effective worker are given as:

**Equation 27**

$$c_2 = \left(\frac{1}{1 + g}\right) * (1 + r) * s; c = c_1 + \left(\frac{1}{1 + n}\right) * c_2$$

Since in the following models we will use three types of production functions, types of interest rates, functional forms and wage rates are given in the following tables

Table 1 Production functions and interest rates

| | $R = R(k)$ |
|---|---|
| CES | $R = \frac{\left(a * \left((b * k^p + 1 - b)^{\frac{1}{p}}\right) * (b * k^p)\right)}{\left(\left(k * (b * k^p + 1 - b)\right) - d\right)}$ |
| Cobb-Douglas | $R = a * b * k^{b-1} - d;$ |
| Other production function | $R = \frac{a}{1 + k} - \frac{a * k}{(1 + k)^2} - d$ |

Table 2 Production functions ,functional form and G-function that should equal to zero to satisfy fundamental difference equation $k_{t+1} = \frac{sw((k_t), r(k_{t+1}))}{1+n}$

| | Functional form $f = f(k)$ | $G = G(k_t, k_t + 1)$ |
|---|---|---|
| CES | $f = a * \left(b * k^p + (1 - b)\right)^{\frac{1}{p}};$ | $G = \frac{S\left(W(k_t), R(k_{t+1})\right)}{1 + n} - k_{t+1}$ |
| Cobb-Douglas | $f = a * k^b;$ | |
| Other production function | $f = b + \left(\frac{a * k}{1 + k}\right)$ | |

Table 3 Production functions and wage rates

| | $W = W(k)$ |
|---|---|
| CES | $W = f(k) - \frac{\left(k * (a * ((b * k^p + 1 - b)^{\frac{1}{p}}) * (b * k^p)\right)}{\left(k * (b * k^p + 1 - b)\right)}$ |
| Cobb-Douglas | $W = f(k) - k * (a * b * k^{b-1});$ |
| Other production function | $W = f(k) - k * (\frac{a}{1 + k} - \frac{a * k}{(1 + k)^2})$ |

Table 4 Production functions and type of instantaneous utility function

| | |
|---|---|
| CRRA | $u(c) = \begin{cases} \ln(c), when\ m = 1 \\ \frac{c^{1-m} - 1}{1 - m} \end{cases}$ |

| | |
|---|---|
| Subsistence consumption | $u(c) = \begin{cases} \ln(c - c_{min})\, when\ m = 1 \\ \dfrac{c - c_{min}{}^{1-m}}{1-m}, c_{min} \geq 0 \end{cases}$ |
| CARA | $u(c) = -exp(-m * c);$ |
| CARA like | $u(c) = \begin{cases} -c^{-1}\, when\ c < 0.773 \\ -exp(-m * c), otherwise \end{cases}$ |

Now, real net rate of return of capital equals:

**Equation 28**

$$r_t = \frac{\hat{r}_t K_t - \delta K_t}{K_t} = \hat{r}_t - \delta$$

Consumption of young and old:

**Equation 29**

$$C_t = c_{1t} L_t + c_{1t} L_{t-1}$$

Saving problem of the young is presented as:

**Equation 30**

$maxU(c_{1t}, c_{2t}) = u(c_{1t}) + (1 + \rho)^{-1} u(c_{2t+1})$ subject to

$$c_{1t} + s_t = w_t$$

Where :

**Equation 31**

$$c_{2t+1} = (1 + r_{t+1}) s_t \,;\, r_{t+1} > -1$$
$$c_{1t} \geq 0 c_{2t+1} \geq 0$$

Identity that holds here is:

**Equation 32**

$$c_{1t} + \frac{1}{1 + r_{t+1}} c_{2t+1} = w_t$$

Where , $\lim_{c \to 0} u'(c) = \infty$ means no fast consumption

Solving the saving problem we got:

**Equation 33**

$$U(c_{1t}, c_{2t+1}) = u(w_t, s_t) + (1 + \rho)^{-1} u\big((1 + r_{t+1}) s_t\big) \equiv \widetilde{U}(s_t)$$

$$\frac{d\widetilde{U}(s_t)}{ds_t} = -u'(w_t - s_t) + (1 + \rho)^{-1} u'\big((1 + r_{t+1}) s_t\big)(1 + r_{t+1}) = 0$$

$$\frac{d^2 \widetilde{U}(s_t)}{ds^2{}_t} = u''(w_t - s_t) + (1 + \rho)^{-1} u''\big((1 + r_{t+1}) s_t\big)(1 + r_{t+1})^2 < 0$$

$$\lim_{s_t \to 0} \frac{d\widetilde{U}(s_t)}{ds_t} = -u'(w_t) + (1 + \rho)^{-1}(1 + r_{t+1}) \lim_{s_t \to 0} u'\big((1 + r_{t+1}) s_t\big) = \infty$$

$$\lim_{s_t \to w_t} \frac{d\widetilde{U}(s_t)}{ds_t} = -\lim_{s_t \to w_t} u'(w_t - s_t) + (1 + \rho)^{-1}(1 + r_{t+1}) u'\big((1 + r_{t+1}) w_t\big) = -\infty$$

The consumption Euler function is presented as :

**Equation 34**

$$u'(c_{1t}) = (1 + \rho)^{-1} u'^{(c_{2t+1})(1 + r_{t+1})} \Rightarrow 1 + r_{t+1} = \frac{u'(c_{1t})}{(1 + \rho)^{-1} u'(c_{2t+1})}$$

Marginal rate of substitution between young and old consumption is given as :

$$MRS_{c_2 c_1} = -\frac{dc_{2t+1}}{dc_{1t}} | U = \widetilde{U} = \frac{u'(c_{1t})}{(1 + \rho)^{-1} u'(c_{2t+1})}$$

Properties of the saving function are:

1. $f(s_t, w_t, r_{t+1}) \equiv -u'(w_t - s_t) + (1 + \rho)^{-1} u'\big((1 + r_{t+1}) s_t\big)(1 + r_{t+1})$
2. $\frac{\partial s_t}{\partial w_t} = \frac{-\partial f(\cdot)/\partial w_t}{D}$
3. $\frac{\partial s_t}{\partial r_{t+1}} = -\frac{\partial f(\cdot)/\partial r_{t+1}}{D}$
4. $D \equiv \frac{\partial f(\cdot)}{\partial s_t} = u''(c_{1t}) + (1 + \rho)^{-1} u''(c_{2t+1})(1 + r_{t+1})^2 < 0$
5. $\frac{\partial f(\cdot)}{\partial w_t} = -u''(c_{1t}) > 0$
6. $\frac{\partial f(\cdot)}{\partial r_{t+1}} = (1 + \rho)^{-1}[u'(c_{2t+1}) + u''(c_{2t+1}) s_t (1 + r_{t+1})]$
7. $s_w \equiv \frac{\partial s_t}{\partial w_t} = \frac{u''(c_{1t})}{D} > 0$

8. $s_r \equiv \frac{\partial s_t}{\partial r_{t+1}} = -\frac{[u'(c_{2t+1})+u''(c_{2t+1})c_{2t+1}]}{D}$
9. $s_{r=}\frac{(1+\rho)^{-1}u'(c_{2t+1})[\theta(c_{2t+1})-1]}{D} \gtreqless 0, for\ \theta(c_{2t+1}) \gtreqless 1$
10. $\theta(c_{2t+1}) \equiv -\frac{c_{2t+1}}{u'(c_{2t+1})}u''(c_{2t+1}) > 0$

Explicit solution of the savings of the young is presented as:
**Equation 35**

$$s_t = \frac{1}{1+(1+\rho)^{\frac{1}{\theta}}(1+r_{t+1})^{\frac{\theta-1}{\theta}}}w_t$$

Elasticity of intertemporal substitution in consumption is determined by following expressions:

1. $\varepsilon_{\left(c_2/_{c_1}\right)} = \frac{MRS}{c_2/c_1}\frac{d(c_2/c_1)}{dMRS}|U = \widetilde{U} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta MRS}{MRS}}$
2. $MRS = -\frac{dc_2}{dc_1}|_{U=\bar{U}} = \frac{u'(c_1)}{\beta u'(c_2)} = 1+r \equiv R$
3. $\sigma(c_1,c_2) = \frac{R}{c_2/c_1}\frac{d(c_2/c_1)}{dR}|_{U=\bar{U}} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta R}{R}}$

Where $\theta(c) \equiv -cu''(c)/u'(c)$ absolute elasticity of marginal utility of consumption :
**Equation 36**

$$\sigma(c_1,c_2) = \frac{c_2+Rc_1}{c_2\theta(c_1)+Rc_1\theta(c_2)}$$

If u(c) belongs to the CRRA class i.e if $\theta(c_1) = \theta(c_2) = \theta$,than $\sigma(c_1,c_2) = 1/\theta$

Clearing in the factor markets we can get the expressions for supply of capital and labor by young and old and distribution of wages:

1. $K_t^d = K_t$
2. $L_t^d = L_t = L_0 + (1+n)^t$
3. $r_t = f'(k_t) - \delta \equiv r(k_t)r' = f''(k_t) < 0$
4. $w_t = f(k_t) - f'(k_t)k_t \equiv w(k_t)w' = -k_t f''(k_t) > 0$

Technically feasible paths of the economy are:

1. $C_t = Y_t - S_t\ = F(K_t, L_t)(K_{t+1} - K_t + \delta K_t)$
2. $C_t \equiv \frac{C_t}{L_t} = \frac{c_{1t}L_t + c_{2t}L_{t-1}}{L_t} = c_{1t} + \frac{c_{2t}}{1+n}$

Equilibrium in the goods market obtains :
**Equation 37**

$$c_{1t}L_t + c_{2t}L_{t-1} + s_tL_t - K_t + \delta K_t = F\left(K_t^d, L_t^d\right)$$

An equilibrium path of the economy is :
**Equation 38**

$$k_{t+1} = \frac{s\big(w(k_t)\big), r(k_{t+1}))}{1+n}$$

First derivative of the previous expression is:
**Equation 39**

$$\frac{dk_{t+1}}{dk_t} = \frac{1}{1+n}\left[s_w(\cdot)w'(k_t) + s_r(\cdot)r'k_{t+1}\frac{dk_{t+1}}{dk_t}\right]$$

The slope of the transition curve can be written as
**Equation 40**

1. $\frac{dk_{t+1}}{dk_t} = \frac{-s_w\big(w(k_t), r(k_{t+1})\big)k_t f''(k_t)}{1+n-s_r\big(w(k_t), r(k_{t+1})\big)f''(k_{t+1})}$
2. $\frac{dk_{t+1}}{dk_t} \lesseqgtr 0, if\ \exists s_r\big(w(k_t), r(k_{t+1})\big) \lesseqgtr \frac{1+n}{f''(k_t)}$

No fast consumption and positive slop assumption prepositions give:

1. $k_{t+1} = \varphi(k_t)$
2. $\varphi'(k_t) = \frac{-s_w\big(w(k_t), r(\varphi(k_t))\big)k_t f''(k_t)}{1+n-s_r\big(w(k_t), r(\varphi(k_t))\big)f''(\varphi(k_t))}$

In the Cobb-Douglas case

1. $u(c) = ln(c)$
2. $Y = AK^aL^{1-a}$

Transition function is
**Equation 41**

$$k_{t+1} = \frac{(1-a)Ak_t^a}{(1+n)(2+\rho)}$$

If the production function is CES type

**Equation 42**

$$f(k) = A\left(\alpha k^{\beta} + 1 - \alpha\right)^{1\backslash\beta}$$

If the elasticity of substitution between capital and labor is $\frac{1}{1-\beta} \gg 0$ i.e.

**Equation 43**

$$\frac{1}{1-\beta} > \frac{1-\left(\frac{1}{1-\beta}\right)}{1+(1+\rho)^{-\rho}(1+f'(k_t)-\delta)^{\rho-1}}$$

The golden rule applies:

**Equation 44**

$$c_t \equiv \frac{C_t}{L_t} = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}$$

In steady state $k_{t+1} = k_t = k$

**Equation 45**

$$c = f(k) - (\delta + n)k \equiv c(k)$$

The first order condition for the previous problem is :

1. $c'(k) = f'(k) - (\delta + n) = 0$
2. $f'(k^G) - \delta = n$

In steady state $f'(k^G) - \delta$ net marginal productivity of capital=growth rate of the economy (n) -highest sustainable level of consumption per unit labor

**Overaccumulation and feasibility and inefficiency**

When does over accumulation occurs ? Following expressions are detailing phenomena:

1. $r^* = f(k) - \delta$ interest rate in steady state
2. $r^* \lesseqgtr f(k) - \delta \gtreqless n \Leftrightarrow k^* \lesseqgtr k^G$

Dynamic efficiency and double infinity provides that : when $\{(c_t, k_t)\}_{t=0}^{\infty}$ feasible path but dynamically inefficient $t \to \infty$ .This leads to:

**Equation 46**

$k_t \to k^* > k^G$ if $k^* > k^G$ ,$\exists \varepsilon > 0, k \in (k^* - 2\varepsilon, k^* + 2\varepsilon) f'(k) - \delta < n$ , by concavity of $f$

$$f(k) - f'(\,k - \varepsilon) \leq f'(k - \varepsilon)\varepsilon$$

And for the consumption:

**Equation 47**

$$\begin{aligned}\hat{c}_t = f(\hat{k}_t) + (1-\delta)\hat{k}_t - (1+n)\hat{k}_{t+1} &= f(k_t - \varepsilon) \\ &+ (1-\delta)(k_t - \varepsilon) \\ &- (1+n)(k_{t+1} - \varepsilon) \\ &> f(k_t) - (\delta + n)\varepsilon \\ &+ (1-\delta)k_t \\ &- (1+n)k_{t+1} + (\delta + n)\varepsilon \\ &= f(k_t) + (1-\delta)k_t \\ &- (1+n)k_{t+1} = c_t\end{aligned}$$

When , $\{(c_t, k_t)\}_{t=0}^{\infty}$ previous expressions are feasible path but dynamically inefficient for $t \to \infty$ and $k_t \to k^* \leq k^G$ .The fact that $k^* > k^G$;and therefore dynamic inefficiency, cannot beruled out might seem to contradict the First Welfare Theorem. This is the theorem saying that when increasing returns to scale are absent, markets are competitive and complete, no goods are of public good character, and there are no other kinds of externalities, then market equilibria are Pareto optimal. In fact, however, the First Welfare Theorem also presupposes a finite number of periods or, if the number of periods is infinite, then a finite number of agents. In contrast, in the OLG model there is a double infinity: an infinite number of periods and agents. Hence, the First Welfare Theorem breaks down. Now, $r^* < n$ and $k^* > k^G$ can arise under laissez faire and by Deriving intertemporal elasticity of substitution one gets:

**Equation 48**

1. $x \equiv \frac{c_2}{c_1} u'(c_1) = \beta u'(xc_1)\, R$
2. $u(c_1) + \beta u(xc_1) = \bar{U}$

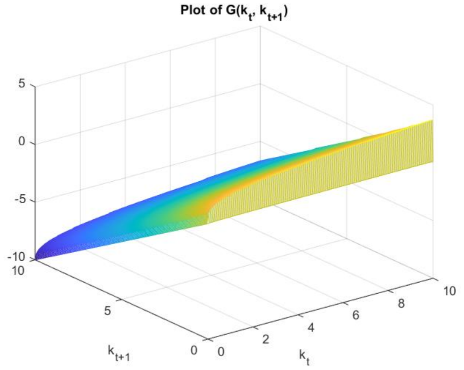
Optimality conditions require :

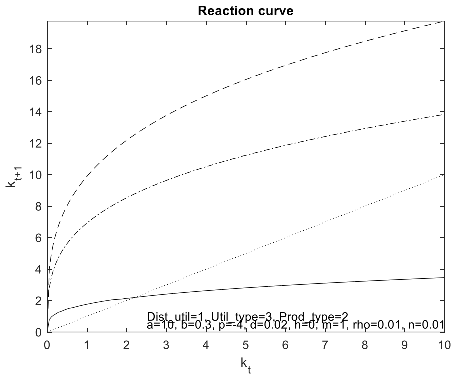
1. $[u''(c_1) - \beta R u''(xc_1)x]c'^{(R)} - \beta R u''(xc_1)c_1 x'(R) = \beta u'(xc_1)$
2. $[u'(c_1) + \beta u'(xc_1)\, R]c'(R) = -\beta u'(xc_1)\, R$
3. $-\left[x\frac{c_1 u''(c_1)}{u'(c_1)} + R\frac{xc_1 u''(xc_1)}{u'(xc_1)}\right]\frac{R}{x}x'(R) = x + R$
4. $\theta(c) \equiv -cu''(c)/u'(c)$
5. $\frac{R}{x}x'(R) = \frac{x+R}{x\theta c_1 + R\theta\,(xc_1)}$

**MATLAB code for OLG model used to visualize the phase space (Klemp,Groth,2009)**

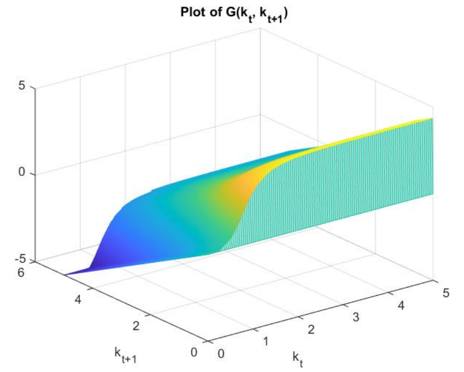
Next are presented three cases of OLG with graphs , i.e. these codes[12] are being used to visualize the reaction curves, by plotting $k_t$ versus $k_{t+1}$. Next are presented graphs for the three cases : Case A –Benchmark case, Case B –Poverty traps, Case C-multiple equilibria case.After the reaction curves and plots parameters used in the three cases are being written in the tables.

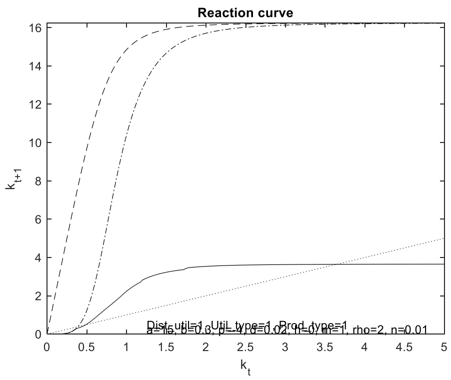
Graphs 1,2 Case A- Benchmark case



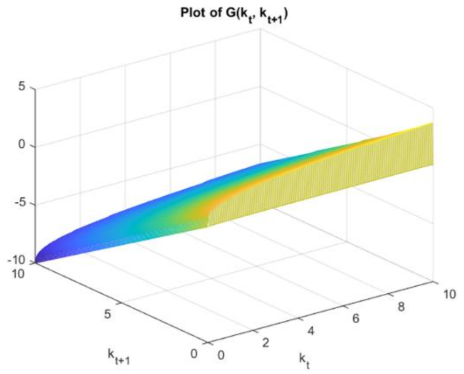



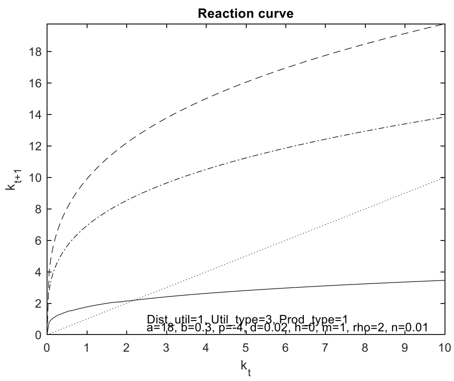

Graphs 3,4 Case B- Poverty traps




Graphs 5,6 Case C- Multiple equilibria




[12] By Marc p. B. Klemp and Christian Groth

Table 5 Parameter values

| Parameters | Case A-benchmark case | Case B-Poverty traps | Case C-Multiple equilibria |
|---|---|---|---|
| Production function | Cobb-Douglas: $a * k^b$ | CES: $a * \left(b * k^p + (1 - b)\right)^{\frac{1}{p}}$ | CES: $a * \left(b * k^p + (1 - b)\right)^{\frac{1}{p}}$ |
| Utility type | CRRA: $log(c)\ when\ m = 1\ and\ (c.^{1-m})/(1 - m)\ when\ m! = 1\ and\ m > 0$ | CRRA: $log(c)\ when\ m = 1\ and\ (c.^{1-m})/(1 - m)\ when\ m! = 1\ and\ m > 0$ | CARA: $-exp(-m * c)\ for\ m > 0$ |
| Parameter of all three production functions-$a$ | $a = 10$ | $a = 15$ | $a = 18$ |
| Parameter of all three production functions-$b$ | $b = 0.3$ | $b = 0.3$ | $b = 0.3$ |
| Parameter of the CES production function-$p$ | $p =/$ | $p = -4$ | $p = -4$ |
| Capital depreciation rate-$d$ | $d = 0.02$ | $d = 0.02$ | $d = 0.02$ |
| Subsistence consumption in the subsistence CRRA utility function-$h$ | $h = 0$ | $h = 0$ | $h = 0$ |
| Parameter of all three utility functions-$m$ | $m = 1$ | $m = 1$ | $m = 5$ |
| Utility discount rate-$\rho$ | $\rho = 0.01$ | $\rho = 2$ | $\rho = 2$ |
| Population growth rate-n | $n = 0.01$ | $n = 0.01$ | $n = 0.01$ |

The previous table 5 shows the combination of parameters used to describe the three economies : A-benchmark case, B-poverty traps, C-multiple equilibria case.

## Conclusion

An overlapping generations models is a representative agent economic model in which agent lives finite periods and, agent sover lap at least one period with another agent's life. OLG models represent a framework to study the allocation of resources across different generations. Two period OLG models can be summarized as follows: Let's suppose that the individual lives in two periods, for this individual budget constraint is set as follows: $c_{1t} + s_{1t} = w_{1t}$; $c_{2(t+1)} + s_{2(t+1)} = w_{2(t+1)}$. In the previous expression $c_{1t}; s_{1t}; w_{1t}$, represent the : consumption, saving and labor income of young population, $c_{2(t+1)}; s_{2(t+1)}; w_{2(t+1)}$, represent the : consumption, saving and labor income of old population. If there are $N_t$ young agents, and $N_{t-1}$ old agent, born one period before, than aggregate demand for consumption would be : $c_t = N_{t-1} \cdot c_{2t} + N_t \cdot c_{1t}$. The aggregate supply of labor is given as: $L_t = N_{t-1} \cdot 1 + N_t \cdot 1$. The aggregate supply of capital is given as : $K_t = N_{t-1} \cdot s_{2t} + N_t \cdot s_{1t}$. Each individual chooses optimal consumption-savings plan to maximize utility subject to budget constraint. The number of overlapping generations in each period $t$ depends on the number of periods each agent lives. One term that applies here is economic birth. This term denounces that the "new" agent is included in the economic calculus of the preexisting agents, Weil (2008). Besides neoclassical growth model, OLG models is the second major workhorse in modern macroeconomics. In this model competitive equilibria can be Pareto suboptimal, outside money may have positive value, there may exist continuum of equilibria. The equilibrium in the OLG models is known as recursive equilibrium[13]. Equilibrium interest rate is very low or very high, (below or above the rate of growth of population) dependent on the fact whether economy is populated with patient or impatient consumers. In the first case $r < n$ equilibrium is not Pareto optimal and in the second case $r > n$, equilibrium is Pareto optimal. Arrival of the "new" agents with the number of dated goods implies that the total number of distinct economic agents is infinite in the overlapping generations model. Previous statement as we said in the introduction of this paper is incompatible with the First welfare theorem, where either the number of periods or number of agents must be finite.

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## Appendix 1 Golden rules and Ramsey exercise

$$\frac{dk}{dt} = sf(k) - nk$$

Or, because $sf(k) > f(k) - c$, then: $\frac{dk}{dt} = f(k) - c - nk$.Thus, we maximize the intertemporal utility stream subject to this equation as a constraint. To solve the problem, we can use the calculus of variations or the maximum principle. Let us use the latter. Thus, setting up the present-value Hamiltonian:

$H = U(ct) + \lambda(f(k) - c - nk)$ where $\lambda$is the current-value "costate" variable. The first order conditions for a maximum, then, yield:

(1) $\frac{dH}{dc} = U_c - \lambda = 0$

(2) $-\frac{dH}{dk} = \frac{dz}{dt} - \rho\lambda = -\lambda(f(k) - n)$

(3) $dH/d\lambda = dk/dt = f(k) - c - nk$

(4) $lim\, \lambda e^{-\lambda t} = 0$

$U_c = \lambda$ (where $U_c = \frac{dU}{dc}$) the marginal utility of consumption at this time period. $\frac{d\lambda}{dt} = U_{cc}(\frac{dc}{dt})$ (where $U_{cc} = d^2U/dc^2$ - the second derivative)$U_{cc}(dc/dt) - pU_c = -U_c(f(k) - n)$or, rearranging: $dc/dt = -[U_c/U_{cc}][f(k) - n - \rho]$ if we had used a so-called CRRA utility function (i.e. $U(c) = \frac{c^{1-e}}{(1-e)c}$where $0 < e < 1$), then the entire term $[U_c/U_{cc}]$ would have been merely $1/e$, and our equation reduced to: $\frac{dc}{dt} = \left(\frac{1}{e}\right)[f(k) - n - \rho]$.The "solution" to the optimization program will be a pair of differential equations - $\frac{dc}{dt}$ just derived, and $\frac{dk}{dt}$ derived from our third condition:

$\frac{dk}{dt} = f(k) - c - nk$. Balanced growth or steady state growth is $f(k) - n - \rho = 0$, $f(k) - c - nk = 0$, where $c^* = f(k^*) - nk^*$,$f(k) = n + \rho$-Golden Utility growth .he present value of future utility gains from individual consumption at any time period t is then: $U(c_t)e^{-\rho t}$

$$U = \int_0^\infty U(c_t)e^{-\rho t}\,dt$$

$f(k) = n$ represents the Golden rule of growth for Allais (1947), Von Neuman (1937) , Robinson (1962)

**Appendix 2 <u>Blanchard (1985)</u> and <u>Yaari (1965)</u> OLG models**

Unlike Ramsey (1928) where economic agent lives infinitely, in Blanchard (1985) and Yaari (1965) economic agent lives from 0 to $t_d$ . Agents utility function is given as :

$\Lambda(t_d) = \int_0^{t_d} U\left(C(t)\right)e^{-\rho t}dt$ ; $\mathrm{E}\Lambda(t_d) = \int_0^{\bar{t}_d} 1 - F(t)U\left(C(t)\right)e^{-\rho t}dt$

Household budget constrain t is given as

$$\frac{dRA(t)}{dt} = r(t)\,RA(t) + y(t) + c(t)$$

$RA(t)$-represents the real assets, $y(t)$-represents the non-interest income, and $c(t)$ -represents the consumptions of agents. In Yaari (1965),Euler equation for agents consumption is given as :

$$\frac{\dot{C}(t)}{C(t)} = \sigma(C(t)[r(t) = \rho - M(t)]$$

$\sigma$ represents the intertemporal elasticity of substitution, $M(t)$ represents the instant probability of death :

$$M(t) = \frac{F(t)}{1 - F(t)}$$

Actuarial note is one method for real assets assurance its revenue is equal to interest rate revenue plus instant probability of death :

$$r^A(t) = r(t) + M(t)$$

Now, Euler equation for agents consumption will become :

$$\frac{\dot{C}(t)}{C(t)} = \sigma(C(t)[r^A(t) - \rho - M(t)] \\ = \sigma(C(t)[r(t) - \rho]$$

Blanchard (1985) assumes that economic agents can live infinitely, henceforth this economic model is named perpetual youth model .In this model $\lim_{\infty \to 0} M^{-1} = \frac{1}{M}$, so as probability of death approaches zero, effective individual horizon is infinite which leads us back to Ramsey (1928). Population growth is given as:

$$\int_{-\infty}^{t} N(t_0, t)dt_0 = N(0)e^{nt}$$

Aggregate welfare constraint is given as a sum of financial and total welfare :

$$\begin{aligned} af_{tot} + h_{wtot} &= K(t) + D(t) + h_{wtot} \\ &= K(t) + D(t) \\ &+ \int_t^\infty \left(w(\bar{t}) - T(\bar{t})\right. \\ &\left.- G(\bar{t})\right)e^{-r^A(t,\bar{t})}dt + \Phi(t) \\ &\Rightarrow \Phi(t) \\ &= D(t) \\ &- \int_t^\infty \left(w(\bar{t}) - T(\bar{t})\right. \\ &\left.- G(\bar{t})\right)e^{-r^A(t,\bar{t})}dt \end{aligned}$$

If $\Phi(t) = 0$ RET holds, otherwise RET fails. In the previous expression $h_{wtot}$ represents the initial human wealth, додека пак $af_{tot}$represents the individual wealth.