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APPLICATION OF MATHEMATICS IN MUSIC COMBINATORICS AND TWELVE-TONE MUSIC

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Abstract. In this research, the connection between some mathematical topics as combinatorics, congruency and algebra groups and their application in music and music production will be emphasized. We will make a demonstration of the mathematical techniques applied in dodecaphony (twelve-tone music) – specific technique of composing music, in which by the help of mathematical matrix, an equal importance is given to all the 12 tones in the chromatic scale.

Key words: combinatorics, congruency, transposition, inversion, retrograding, twelve-tone music

Introduction

The application of mathematics in other sciences and disciplines is enormous. A lot of people ask them, and are probably wondering if there is mathematics in the music or music in the mathematics. Sounds interesting, doesn’t it? But it’s not simple. Mathematics is everywhere, obvious or not. The researches have shown that although seemingly different, these two disciplines are connected for more than two thousand years. The music is really very mathematical, and the mathematics is characteristically for lot of basic ideas in the theory of music (See: [2], [3], [4], [7]). The music theoreticians, as experts in other disciplines, used mathematics to develop, express and transfer their ideas. In this research, will be emphasized the connection between some mathematical topics as combinatorics, congruency and algebra groups and their application in music and music production. We will make a demonstration of the mathematical techniques applied in the twelve-tone music - a specific technique of composing music, in which by the help of mathematical matrix, equally important is given to all the 12 tones in the chromatic scale. Also, the reader can look at it [1], [4] [5]

Combinatorics

Let there be a finite set $A_n = \{a_1, a_2, a_3, \dots, a_n\}$ whose elements could be persons, objects, plants, animals, numbers, signs, events. From the given finite set, variously ordered sets and subsets can be formed in different ways. In some of these sets, some elements can be repeated i.e. can show several times. For one set $A_n =$

$\{a_1, a_2, a_3, \dots, a_n\}$ we say that is finite, if it is empty or if there is natural number n, so that to each element of the set A one unique element of the set $\{1,2,3,\dots,n\}$ can be attached.

Part of the mathematics that studies the formation of these sets and subsets and the possible order and line of the elements is called combinatorics.
Every ordering of the elements of one finite set in which no element repeats is called permutation without repetition.

The number of permutations without repetition of n elements is $P(n) = n!$

Transposition, inversion, retrograding

Definition 2.1: Let n be an integer by module 12. The function $T_n: Z_{12} \to Z_{12}$, given with

$$T_n(x) = x + n \ (\mathrm{mod} 12)$$

is called **transposition** for n.

Example:

$$T_4: Z_{12} \to Z_{12}$$

$$T_4(5) = 5 + 4 = 9$$

$$T_4(8) = 8 + 4 = 0$$

$$T_4(10) = 10 + 4 = 2$$

For example, the transposition of the accord $\{0,4,7\}$ for 7 steps is $T_7\{0,4,7\} = \{7,11,2\}$. In other words, the transposition of the sequences of note classes x for n semitones is the sequences $T^n(x)$, in which, each tone of that sequences is moved for n semitones. For example if

x = 3 0 8 we get $T^4(x)$ = 7 4 0.

Mathematically, the translation is in correspondence with the transposition. When we have repetition of the same sequences of note classes, it is regarded as a horizontal translation, and if we have vertical movement of the tones, in fact movement for several semitones, then it is a vertical translation.

Definition 2.2: Let n be an integer by module 12. The function $I_n: Z_{12} \to Z_{12}$, given with

$$I_n(x) = -x + n(\mathrm{mod} 12)$$

is called **inversion** for n.

Example: Let

$$I_5: Z_{12} \to Z_{12}$$
$$I_5(4) = -4 + 5 = 1$$

$$I_5(8) = -8 + 5 = -3 = 9$$

$$I_5(10) = -10 + 5 = -5 = 7$$

For example, the inversion of the segment $\{0,0,4,4,7,7,4,5,5,2,2,11,11,7\}$ around 0 would be

$$I_0\{0,0,4,4,7,7,4,5,5,2,2,11,11,7\} = \{0,0,8,8,5,5,7,7,10,10,1,1,5\}$$

The inversion I(x) of the sequences x transforms it in sequences, in which each member n from the sequences x replicates in 12-n.

For example for x = 3 0 8, I(x) = 9 0 4.

The sequences $T^nI(x)$ is considered as an inversion of x as well. For x = 3 0 8, we get $T^6I(x)$ = 3 6 10. The inversion in music is in correlation with the horizontal or vertical symmetry in mathematics. The axis of symmetry in the horizontal symmetry will be the axis which goes through every tone of the sequences or the line of the staff, on the other hand, the axis at the vertical symmetry goes through the tact line or through some tone from the sequences.

Definition 2.3: Retrograding R(x) for the sequences x is the sequences consisting of the members of x, in reverse order.

For x = 3 0 8, R(x) = 8 0 3.

For the operators T,I and R the following relations are given:

$T^{12} = e$, $T^nR = RT^n$, $T^nI = IT^{-n}$, $RI = IR$, where e is an identical operator, it doesn't change the tone or T^0. In the theory of groups, the operation $T^n (0 \leq n \leq 11)$ forms the cyclic group Z_{12}. The operation R, together with the identical operator, forms the cyclic group Z_2. The operations T and R are commutative. [4]

That means that we can say that there are 4 forms of the tone string. The basic form is the original form of the sequences of tones, transposition $T^n(x)$, inversion $T^nI(x)$ and retrograding form $T^nR(x)$. Finally, here we can add the retrograding inversion $T^nRI(x)$.

Congruency and musical scales

For easier mathematical analysis of the musical compositions, there was a need of "translation" of the notes into mathematical language, i.e. correspondence between the pitch classes and the integers. The following "equation" of pitch classes and the numbers:

C=0

C#=D♭=1

D=2

D#=E♭=3

E=4

F=5

F#=G♭=6

G=7

G#=A♭=8

A=9

A#=B♭=10

This means, for example, when we discuss about the accord {C, E, G}, actually we analyze the set {0 ,4, 7}. If we talk about the theme consisting of the following tones

$$\langle C, C, E, E, G, G, E, F, F, D, D, B, B, G \rangle,$$

that mathematically written with set of numbers is $\langle 0,0,4,4,7,7,4,5,5,2,2,11,11,7 \rangle$ These brackets very often are used in the music writings in order to emphasize the subordination of the notes. [2]

Congruency by module 12 is a segment of musical theory, where we use the numbers from 0 to 11 $(12 \equiv 0 (\mathrm{mod} 12))$. Choosing these elements will be presented with the help of the table:

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

From this table we can see the addition in this set is closed operation, so the set with these elements and the operation addition present a group and that is abelian (commutative) group. [4] This group will be marked Z_{12}, 0 is a neutral element, inverse of

$$i, 0 \leq i \leq 11 \text{ is the element } 12 - i.$$

Definition 4.1: Group $(G,\cdot)$ is set G, together with the operation„·“, if the following conditions a fulfilled:

i) G is closed above the operation „·“, i.e. $a \cdot b \in G), (\forall a, b \in G)$

ii) The operation „·“ is associative, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c), (\forall a, b, c \in G)$

iii) There is a neutral element $e \in G$, than $e \cdot a = a \cdot e, (\forall a \in G)$

iv) Each element $a \in G$ has inverse element a^{-1}, than $a^{-1} \cdot a = a \cdot a^{-1} = e$

Twelve-tone music

At the beginning of the XX century, Arnold Shoenberg (1874-1951) created the twelve-tone technique for composing, method in which all 12 tones of the chromatic scale are equal to their appearance in the composition. This way of composing music, after its founding in the first half of the XX century, was continued by his students Anton Webern (1883-1945) Alban Berg (1885-1935), while in the 50's of the XX century this technique of composing became very popular and widely used by Milton Babbitt (1916-2011), Luciano Berio (1925-2003), Pierre Boulez (1925-2016)…

In this method of creating music, there is no scales or tonal center and the consonance is nearly abandoned in favor of the combinatorics [7]. This presents one type of musical serialism which is construed of classes of notes, which, by forming tone series, create unique melodic and harmonic structures. The composition created with this 12-tone system, is made with the help of the row chart with 12 rows and 12 columns, where they have the following characteristics: firstly, each row and each column consists of all 12 notes only once, without repetition [7]. The first row is called basic row or prime row. It is filled by all 12 tones appearing precisely once, which means that it can be whichever sequences of 12 different tones with free order. The number of possible basic rows is $12! = 12 \cdot 11 \cdot 10 \cdot ... \cdot 3 \cdot 2 \cdot 1$, which is equal to 479 001 600 unique row forms. So, that is why great opportunities for composition appear.

The first column consists the sequences of notes received by inversion of the first row. That means that the inverse of the second row is the second column, i.e. to the i-row inverse is the i-column, for $0 \leq i \leq 11$. The rows are created with the help of transposition, which is construed with the help of the two elements that are on the position (1,1) and (2,1). From this, we can determine what type of transposition it is for the second row, so from (1,1) and (3,1) for the transposition of the third row etc. from (1,1) and (I,1) for the i-row, for $0 \leq i \leq 11$. When the table is filled in this way, it can be noticed that the rows read from right to left are retrograding sequences, according to each row, and the columns read from down to up are retrograding inversion.

For example, let the sequences {C, A, G, D#, E, F, D, B, B♭, G#, C#, F#} be a substitute for the basic sequences of tones. We fill in the table according to the explained rule:

	I^0	I^9	I^7	I^3	I^4	I^5	I^2	I^{11}	I^{10}	I^8	I^1	I^6	
T^0	C	A	G	D#	E	F	D	B	B♭	G#	C#	F#	R^0
T^3	D#	C	B♭	F#	G	G#	F	D	C#	B	E	A	R^3
T^5	F	D	C	G#	A	B♭	G	E	D#	C#	F#	B	R^5
T^9	A	F#	E	C	C#	D	B	G#	G	F	B♭	D#	R^9
T^8	G#	F	D#	B	C	C#	B♭	G	F#	E	A	D	R^8
T^7	G	E	D	B♭	B	C	A	F#	F	D#	G#	C#	R^7
T^{10}	B♭	G	F	C#	D	D#	C	A	G#	F#	B	E	R^{10}
T^1	C#	B♭	G#	E	F	F#	D#	C	B	A	D	G	R^1
T^2	D	B	A	F	F#	G	E	C#	C	B♭	D#	G#	R^2
T^4	E	C#	B	G	G#	A	F#	D#	D	C	F	B♭	R^4
T^{11}	B	G#	F#	D	D#	E	C#	B♭	A	G	C	F	R^{11}
T^6	F#	D#	C#	A	B♭	B	G#	F	E	D	G	C	R^6
	RI^0	RI^9	RI^7	RI^3	RI^4	RI^5	RI^2	RI^{11}	RI^{10}	RI^8	RI^1	RI^6	

Equivalent set to the basic sequences of tones from the example is the set

$$\{ 0, 9, 7, 3, 4, 5, 2, 11, 10, 8, 1, 6 \}.$$

This sequence is the first row in the table. The first column is created by its inversion. Hence, the elements from the first column will be the elements from the following set

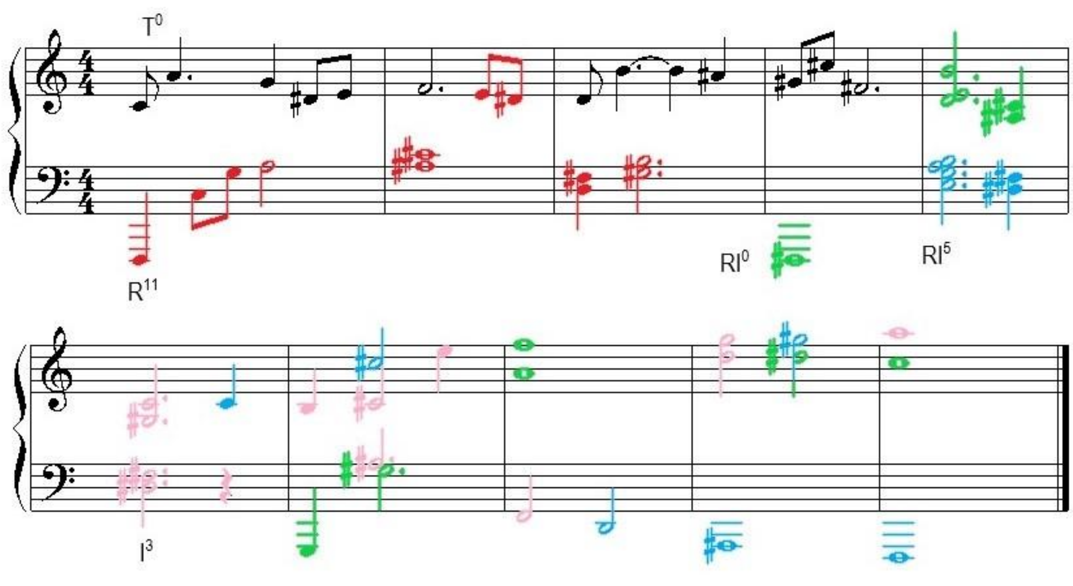
$$\{ 0, 3, 5, 9, 8, 7, 10, 1, 2, 4, 11, 6 \}.$$

From the fact that on position (1,1) is to tone C and on position (2,1) is the tone D#, we get that the second row is a transposition for 3 semitones because D # =3, and C = 0, we have $0 + n = 3 \pmod{12}$, from where we get that n=3. Now, we order the following tones in the second row, with transposition of each tone, accordingly. Similar to this, from the tone F = 5 on position (3,1), we get that the third row is created with translation for 5 semitones, because from $0 + n = 5 \pmod{12}$, we get that n = 5. The procedure is repeated to rows until filling all table. After filling of the whole table, it can be seen that each column is really the inversion of the according row, the first column to the first row, the second column is the inverse to the second row etc. In the same way, it is noticed that if we read the rows from right to left they a retrograding inversions to the rows and columns, reading them from down to up they are retrograding inversions of the according rows.

Analogically to the table, filled with tones, we have the table filled with numbers, from where we can clearly see that it in fact is an algebra group.

	I^0	I^9	I^7	I^3	I^4	I^5	I^2	I^{11}	I^{10}	I^8	I^1	I^6	
T^0	0	9	7	3	4	5	2	11	10	8	1	6	R^0
T^3	3	0	10	6	7	8	5	2	1	11	4	9	R^3
T^5	5	2	0	8	9	10	7	4	3	1	6	11	R^5
T^9	9	6	4	0	1	2	11	8	7	5	10	3	R^9
T^8	8	5	3	11	0	1	10	7	6	4	9	2	R^8
T^7	7	4	2	10	11	0	9	6	5	3	8	1	R^7
T^{10}	10	7	5	1	2	3	0	9	8	6	11	4	R^{10}
T^1	1	10	8	4	5	6	3	0	11	9	2	7	R^1
T^2	2	11	9	5	6	7	4	1	0	10	3	8	R^2
T^4	4	1	11	7	8	9	6	3	2	0	5	10	R^4
T^{11}	11	8	6	2	3	4	1	10	9	7	0	5	R^{11}
T^6	6	3	1	9	10	11	8	5	4	2	7	0	R^6
	RI^0	RI^9	RI^7	RI^3	RI^4	RI^5	RI^2	RI^{11}	RI^{10}	RI^8	RI^1	RI^6	

This is an example of a short melody with combination of sequences of tones from a few rows and columns:



In this example we can see that besides the basic sequence of tones (from the original row), with which the main melody begins, there are a few other rows and columns. It means that besides T^0, the 11th row R^{11} is taken, retrograding of the transposition T^{11}, the first column RI^0 which is a retrograding inversion of the basic (first) row, the sixth column RI^5, retrograding inversion of the sixth row and the fourth column I^3 inversion of the fourth row.

Conclusion

We can conclude that there is an application of mathematics in music. Mathematics helps in creation of music and in listening to music. With this research we show that by applying of mathematical combination of sequences of tones we can create music, which will sound beautifully and the same is very pleasant for listening. It shows that the transposition, inversion and retrograding in the music have some mathematical characteristics. We could say that mathematics as a science completes the music as art and they are in direct correlation.

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