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Simple sufficient conditions for bounded turning

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ABSTRACT - Let f be an analytic function in the open unit disk normalized such that $f(0) = f'(0) - 1 = 0$. In this paper the modulus and the real part of the linear combination of $f'(z)$ and $f(z)/z$ is studied and conditions when f is of bounded turning are obtained.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ that are normalized such that $f(0) = f'(0) - 1 = 0$, i.e. $f(z) = z + a_2z^2 + a_3z^3 + \cdots$.

A function $f \in \mathcal{A}$ is in the class of *starlike functions*, S^*, if and only if

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathbb{D}.$$

Such functions are univalent and their geometric characterization (which motivates the name of the class) is that they map the unit disk onto a starlike region, i.e. if $\omega \in f(\mathbb{D})$ then $t\omega \in f(\mathbb{D})$ for all $t \in [0, 1]$.

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Another well-known class of univalent functions is the *class of functions with bounded turning*,

$$R = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathbb{D}\}.$$

More details on these classes can be found in [2] (section 2.5) and [3] (section 5.5). One of the main results concerning them is due to Krzyz ([7]), who proved that S^* does not contain R and R does not contain S^*. This makes the class R interesting and lots of research is dedicated to it. Some references in that direction are [6] – [9].

In this paper we will study the linear combination of two simple expressions, $f'(z)$ and $f(z)/z$, i.e. we will study the modulus and the real part of

$$\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \tag{1}$$

and obtain criteria for a function $f \in \mathcal{A}$ to be of bounded turning. For that purpose we will use a method from the theory of differential subordinations. Valuable references on this topic are [1] and [3].

First we introduce subordination. Let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disc $\mathbb{D}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. In particular, if $g(z)$ is univalent in $\mathbb{D}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

For obtaining our main result we will use the method of differential subordinations. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [4] and [5]. Namely, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ (where $\mathbb{C}$ is the complex plane) is analytic in a domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{2}$$

A univalent function $q(z)$ is said to be a *dominant* of the differential subordination (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). If $\widetilde{q}(z)$ is a dominant of (2) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (2), then we say that $\widetilde{q}(z)$ is the *best dominant* of the differential subordination (2).

From the theory of first-order differential subordinations we will make use of the following lemma.

LEMMA 1.1 [[5]]. *Let $q(z)$ be univalent in the unit disk $\mathbb{D}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

i) $Q(z)$ is starlike in the unit disk $\mathbb{D}$; and

ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} \left(= \operatorname{Re} \left\{ \dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)} \right\} \right) > 0,\ z \in \mathbb{D}.$

If $p(z)$ is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{3}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (3).

Now, using Lemma 1.1 we will prove the following result.

LEMMA 1.2. *Let $f \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ be such that $\alpha + \beta = 0$ or $\alpha + \beta = 1$. Also, let $q(z)$ be univalent in the unit disk $\mathbb{D}$ with $q(0) = 0$ and*

$$\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D}. \tag{4}$$

Additionally, in the case when $\alpha + \beta = 1$, let $\operatorname{Re} \dfrac{1}{\alpha} > -1$ and

$$\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > -\operatorname{Re} \frac{1}{\alpha}, \quad z \in \mathbb{D}. \tag{5}$$

If

$$\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \prec (\alpha + \beta) \cdot [q(z) + 1] + \alpha z q'(z) (\equiv h(z)) \tag{6}$$

then $\dfrac{f(z)}{z} - 1 \prec q(z)$, and $q(z)$ is the best dominant of (6).

PROOF. The functions $\theta(\omega) = (\alpha + \beta) \cdot (\omega + 1)$ and $\phi(\omega) = \alpha$ are analytic in the domain $D = \mathbb{C}$ which contains $q(\mathbb{D})$ and $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Further, $Q(z) = zq'(z)\phi(q(z)) = \alpha z q'(z)$ is starlike since

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D};$$

furthermore for the function $h(z) = \theta(q(z)) + Q(z)$ we have

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left[1 + \frac{\alpha + \beta}{\alpha} + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D},$$

for $\alpha + \beta = 0$ due to (4) and for $\alpha + \beta = 1$ due to (5).

Now, let choose $p(z) = \frac{f(z)}{z} - 1$ which is analytic in $\mathbb{D}$, $p(0) = q(0) = 0$ and $p(\mathbb{D}) \subseteq D = \mathbb{C}$. Finally, bearing in mind that subordinations (3) and (6) are equivalent, from Lemma 1.1 we deduce the conclusions of Lemma 1.2. $\square$

2. Results on the modulus of (1)

In this section we will study the modulus of (1) and obtain conclusions that lead to criteria for a function f to be in the class R.

THEOREM 2.1. *Let $f \in \mathcal{A}$, $\mu > 0$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ be such that $\alpha + \beta = 0$ or $\alpha + \beta = 1$. Also, when $\alpha + \beta = 1$ let $\operatorname{Re} \frac{1}{\alpha} > -1$. If*

$$\left|\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta)\right| < \delta \equiv \begin{cases} \mu \cdot |\alpha|, & \alpha + \beta = 0 \\ \mu \cdot |1 + \alpha|, & \alpha + \beta = 1 \end{cases}, \tag{7}$$

for all $z \in \mathbb{D}$, then

$$\left|\frac{f(z)}{z} - 1\right| < \mu, \quad z \in \mathbb{D}. \tag{8}$$

This implication is sharp, i.e., in the inequality (8) μ cannot be replaced by a smaller number for the implication to hold. Also,

$$|f'(z) - 1| < \lambda \equiv \left\{ \begin{array}{cc} 2\mu, & \alpha + \beta = 0 \\ \mu \cdot \left(\left|1 + \frac{1}{\alpha}\right| + \left|1 - \frac{1}{\alpha}\right|\right), & \alpha + \beta = 1 \end{array} \right\}, \quad z \in \mathbb{D}.$$

This implication is also sharp, i.e., λ cannot be replaced by a smaller number for the implication to hold, if

(i) $\alpha + \beta = 0$; *or*
(ii) $\alpha + \beta = 1$ *and* $\left|1 + \frac{1}{\alpha}\right| + \left|1 - \frac{1}{\alpha}\right| = 2$.

Additionally, if $\mu \leq \frac{1}{2}$ for $\alpha + \beta = 0$ or $\left|1 + \frac{1}{\alpha}\right| + \left|1 - \frac{1}{\alpha}\right| \leq \frac{1}{\mu}$ for $\alpha + \beta = 1$ then $f \in R$.

PROOF. Choosing $q(z) = \mu z$ we have $1 + \frac{zq''(z)}{q'(z)} = 1$, meaning that (4) and (5) from Lemma 1.2 both hold. Further, for the function $h(z)$ defined in (6) we have

$$h(z) = \alpha + \beta + \mu z(2\alpha + \beta),$$

meaning that the subordination (6) is equivalent to

$$\left|\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta)\right| < \mu \cdot |2\alpha + \beta| = \delta, \quad z \in \mathbb{D},$$

and so equivalent to (7). Therefore, (8) follows directly from Lemma 1.2 and the definition of subordination.

Further, for all $z \in \mathbb{D}$,

$$\left|\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta)\right| = \left|\alpha \cdot [f'(z) - 1] + \beta \cdot \left[\frac{f(z)}{z} - 1\right]\right|$$

and

$$|\alpha| \cdot |f'(z) - 1| \le \left|\alpha \cdot [f'(z) - 1] + \beta \cdot \left[\frac{f(z)}{z} - 1\right]\right| + \left|\beta \cdot \left[\frac{f(z)}{z} - 1\right]\right|$$
$$< \delta + |\beta| \cdot \mu = |\alpha| \cdot \lambda,$$

since $|w_1| \le |w_1 + w_2| + |w_2|$. Therefore, the implications of this corollary holds.

Both implications are sharp as the function $f_*(z) = z + \mu z^2$ shows, since

$$\left|\alpha \cdot f_*'(z) + \beta \cdot \frac{f_*(z)}{z} - (\alpha + \beta)\right| = \mu \cdot |2\alpha + \beta| \cdot |z| = \delta \cdot |z|, \quad z \in \mathbb{D},$$

$$\left|\frac{f_*(z)}{z} - 1\right| = \mu \cdot |z|, \quad z \in \mathbb{D},$$

$$|f_*'(z) - 1| = 2 \cdot \mu \cdot |z|, \quad z \in \mathbb{D},$$

and $2\mu = \lambda$ if (i) or (ii) hold. □

3. Results on the real part of (1)

In this section we study the real part of the expression (1) and obtain criteria that ensure that a function $f \in \mathcal{A}$ is in the class R.

THEOREM 3.1. *Let $f \in \mathcal{A}$, $\mu > 0$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ be such that $\alpha + \beta = 0$ or $\alpha + \beta = 1$. Also, when $\alpha + \beta = 1$ let $\operatorname{Re} \frac{1}{\alpha} > 0$. If*

$$\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \prec (\alpha + \beta)\left(1 + \frac{2\mu z}{1 - z}\right) + \frac{2\alpha\mu z}{(1 - z)^2} \equiv h_2(z) \tag{9}$$

then

$$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1-\mu, \quad z \in \mathbb{D}. \tag{10}$$

This implication is sharp, i.e., in the inequality (10) μ *cannot be replaced by a bigger number so that the implication still holds.*

PROOF. The implication of this theorem follows directly from Lemma 1.2 for $q(z) = \frac{2\mu z}{1-z}$. Condition $\operatorname{Re}\frac{1}{\alpha} > 0$ stands in place of $\operatorname{Re}\frac{1}{\alpha} > -1$ in order for (5) to hold. The result is sharp, as can be seen from the function $f_*(z) = z + z \cdot q(z)$ for which

$$\alpha \cdot f_*'(z) + \beta \cdot \frac{f_*(z)}{z} = (\alpha+\beta)\left(1+\frac{2\mu z}{1-z}\right) + \frac{2\alpha\mu z}{(1-z)^2}$$

and $\operatorname{Re}\frac{f(z)}{z} = 1-\mu$ for $z = -1$.

In the case when $\alpha + \beta = 1$ we obtain the following corollary.

COROLLARY 3.2. *Let* $f \in \mathcal{A}$, $\alpha > 0$ *and* $\mu > 0$. *If*

$$\operatorname{Re}\left[\alpha \cdot f'(z) + (1-\alpha)\cdot\frac{f(z)}{z}\right] > 1 - \mu\cdot\left(1+\frac{\alpha}{2}\right), \quad z \in \mathbb{D}, \tag{11}$$

then

$$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1-\mu, \quad z \in \mathbb{D}.$$

If, additionally,

(i) $\alpha > 1$ *and* $\mu \le 1$; *or*
(ii) $\alpha < 1$ *and* $\mu \ge 1$;

then

$$\operatorname{Re} f'(z) > 1 - \frac{3}{2}\cdot\mu, \quad z \in \mathbb{D}. \tag{12}$$

These results are sharp.

PROOF. Let $\alpha + \beta = 1$. Then, for the function h_2 defined in (9) we have

$$h_2(z) = 1 + \frac{2\mu z}{1-z} + \frac{2\alpha\mu z}{(1-z)^2},$$

$h_2(0) = 1$ and

$$h_2(e^{i\theta}) = 1 - \frac{\mu\alpha}{2}(1+t^2) - \mu + \mu t i,$$

where $t = \operatorname{ctg}(\theta/2)$. Therefore,

$$X = \operatorname{Re} h(e^{i\theta}) = 1 - \mu\left(\frac{\alpha}{2}+1\right) - \frac{\alpha}{2\mu}\cdot Y^2,$$

where

$$Y = \operatorname{Im} h(e^{i\theta}) = \mu t$$

attains all real numbers. This leads to

$$h_2(e^{i\theta}) = \left\{x + iy : x = 1 - \mu\left(1+\frac{\alpha}{2}\right) - \frac{\alpha}{2\mu}\cdot y^2, y \in \mathbb{R}\right\}.$$

From here, bearing in mind the definition of subordination, the inequality (11) and the fact that

$$\left\{x + iy : x > 1 - \mu\left(1+\frac{\alpha}{2}\right), y \in \mathbb{R}\right\} \subseteq h_2(\mathbb{D}),$$

we obtain the subordination (9). Therefore, from Theorem 3.1 it follows that

$$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1 - \mu, \quad z \in \mathbb{D}.$$

Further, in the case when (i) or (ii) holds we have

$$\begin{aligned}\operatorname{Re} f'(z) &= \frac{1}{\alpha}\cdot\left\{\operatorname{Re}\left[\alpha\cdot f'(z) + (1-\alpha)\cdot\frac{f(z)}{z}\right] - (1-\alpha)\cdot\operatorname{Re}\left[\frac{f(z)}{z}\right]\right\}\\ &> \frac{1}{\alpha}\cdot\left[1 - \mu\left(1+\frac{\alpha}{2}\right) - (1-\alpha)(1-\mu)\right] = 1 - \frac{3}{2}\cdot\mu,\end{aligned}$$

for all $z \in \mathbb{D}$.

The results are sharp in view of the function $f_*(z) = z + \dfrac{2\mu z^2}{1-z}$ for which $f_*(z)/z = 1 + \dfrac{2\mu z}{1-z} \equiv g(z)$, $g(\mathbb{D}) = \{x + iy : x > 1 - \mu, y \in \mathbb{R}\}$,

$$\alpha\cdot f_*'(z) + (1-\alpha)\cdot\frac{f_*(z)}{z} = h_2(z)$$

and

$$\operatorname{Re} f_*'(z) = \operatorname{Re} h_2(z) = 1 - \frac{3}{2}\cdot\mu \quad \text{for} \quad z = -1.$$

□

In a similar way to Corollary 3.2, for the case $\alpha = -\beta = 1$ we obtain

COROLLARY 3.3. *Let $f \in \mathcal{A}$ and $\mu > 0$. If*

$$\operatorname{Re}\left[f'(z) - \frac{f(z)}{z}\right] > -\frac{\mu}{2}, \quad z \in \mathbb{D}, \tag{13}$$

then $\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1 - \mu$, $z \in \mathbb{D}$, *and* $\operatorname{Re} f'(z) > 1 - \frac{3}{2} \cdot \mu$, $z \in \mathbb{D}$. *If, additionally,* $\mu \leq \frac{2}{3}$, *then* $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$, *i.e.* $f \in R$. *Both implications are sharp.*

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