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SOME NEW FIXED POINT THEOREMS IN 2-BANACH SPACES

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Abstract. S. Ghler ([9]), 1965, defined the 2-normed space, A. White ([3]), 1968, defined the 2-Banach space. Several statements about them are proven in [7]. P. K. Hatikrishnan and K. T. Ravindran in [5] defined the contractive mapping in 2-normed space. M. Kir and H. Kiziltunc in [3] by applying the above theorem, proved the generalizations of R. Kannan ([6]) and S. K. Chatterjea ([10]) theorem. Further generalizations of these results are elaborated in [1] and [11]. In this paper we will generalize the above results by using the class Θ of monotony increasing functions $f:[0,+\infty)\to\mathbf{R}$ such that $f^{-1}(0)=\{0\}$ holds true.

1. INTRODUCTION

Theory of fixed point is rapidly developing last decades. S. Ghler ([9]) and A. White ([2]), 1965 and 1969, defined the 2-normed and 2-Banach spaces, and certain classical results about this theory, are generalized in 2-normed and 2-Banach spaces. P. K. Hatikrishnan and K. T. Ravindran defined contractive mapping in 2-normed space.

Definition 1. ([5]). Let $(L,||\cdot,\cdot||)$ be a real 2-normed space. Mapping $T:L\to L$ is said to be contraction if there exists $\lambda\in[0,1)$ such that $||Tx-Ty,z||\leq\lambda||x-y,z||$, for all $x,y,z\in L$, holds true.

P. K. Hatikrishnan and K. T. Ravindran in [5] proved that for each contractive mapping, a unique fixed point in closed and bounded subset of 2-Banach space L, exists.

M. Kir and H. Kiziltunc in [3] proved that if $S:L\to L$ satisfies one of the following conditions

$$||Sx-Sy,z||\leq\lambda(||x-Sx,z||+||y-Sy,z||) \tag{1}$$

for all $x,y,z\in L$

or

$$||Sx-Sy,z||\leq\lambda(||x-Sy,z||+||y-Sx,z||) \tag{2}$$

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for all $x, y, z \in L$ for $\lambda \in (0, \frac{1}{2})$, then it exists a unique fixed point for S in L. In [11] further generalizations for these results are proven, and in [1], by using a sequentially convergent mapping, are proved the generalized M. Kir and H. Kiziltunc theorems. In our further considerations, by using the class Θ of monotony increasing continuous functions $f : [0, +\infty) \to \mathbf{R}$ such that $f^{-1}(0) = \{0\}$, we will elaborate the generalizations for some of the stated results. It is necessary to be said that if $f \in \Theta$, then $f^{-1}(0) = \{0\}$ implies that $f(t) > 0$, for each $t > 0$, holds true, and that the proves will be done by using the sequentially convergent, defined as the following:

Definition 2. Let $(L, ||\cdot, \cdot||)$ be a 2-normed space. A mapping $T : L \to L$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence.

2. MAINS RESULTS

Theorem 1. Let $(L, ||\cdot, \cdot||)$ be a 2-Banach space, $S : L \to L$, $f \in \Theta$ and $T : L \to L$ is continuous, injection and sequentially convergent mapping. If there exists $\alpha \in (0, \frac{1}{2})$ such that

$$f(||TSx - TSy, z||) \leq \alpha[f(||Tx - TSx, z||) + f(||Ty - TSy, z||)] \quad (3)$$

for all $x, y, z \in L$, is satisfied, then for S it exists a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the above fixed point.

Proof. Let x_0 be any point on L and the sequence $\{x_n\}$ be defined as the following $x_{n+1} = Sx_n$, $n = 0, 1, 2, 3, ...$ The inequality (3) implies

$$\begin{aligned} f(||Tx_{n+1} - Tx_n, z||) &= f(||TSx_n - TSx_{n-1}, z||) \\ &\leq \alpha[f(||Tx_n - TSx_n, z||) + f(||Tx_{n-1} - TSx_{n-1}, z||)] \\ &\leq \alpha[f(||Tx_n - Tx_{n+1}, z||) + f(||Tx_{n-1} - Tx_n, z||)] \end{aligned}$$

So,

$$f(||Tx_{n+1} - Tx_n, z||) \ \leq \lambda f(||Tx_n - Tx_{n-1}, z||) \quad (4)$$

for each $n = 0, 1, 2, 3, ...$ and each $z \in L$, for $\lambda = \frac{\alpha}{1-\alpha} < 1$. The inequality (4) implies

$$f(||Tx_{n+1} - Tx_n, z|| \ ) \leq \lambda^n f(||Tx_1 - Tx_0, z||) \quad (5)$$

for each $n = 0, 1, 2, 3, ...$and each $z \in L$. Further, (5) implies that for all $m, n \in \mathbf{N}$, $n > m$ and each $z \in L$

$$\begin{aligned} f(||Tx_n - Tx_m, z||) &= f(||TSx_{n-1} - TSx_{m-1}, z||) \\ &\leq \alpha[f(||TSx_{n-1} - Tx_{n-1}, z||) + f(||TSx_{m-1} - Tx_{m-1}, z||)] \\ &= \alpha[f(||Tx_n - Tx_{n-1}, z||) + f(||Tx_m - Tx_{m-1}, z||)] \\ &\leq \alpha[\lambda^{n-1} f(||Tx_1 - Tx_0, z||) + \lambda^{m-1} f(||Tx_1 - Tx_0, z||)]. \end{aligned}$$

holds true. The latter implies that

$$\lim_{m,n\to\infty} f(||Tx_n - Tx_m, z||) = 0$$

, for each $z \in L$. And since $f \in \Theta$ we get that

$$\lim_{m,n\to\infty} ||Tx_n - Tx_m, z|| = 0$$

, for each $z \in L$. Thus, the sequence $\{Tx_n\}$ is Caushy, and since L is 2-Banach space, the sequence is convergent. Further, the mapping $T : L \to L$ is sequentially convergent, and therefore the sequence $\{x_n\}$ is convergent, i.e. it exists $u \in L$ such that $\lim_{n\to\infty} x_n = u$, holds true. The continuous of T implies that $\lim_{n\to\infty} Tx_n = Tu$. So, for each $z \in L$

$$\begin{aligned} f(||TSu - Tx_{n+1}, z||) &= f(||TSu - TSx_n, z||) \\ &\leq \alpha[f(||TSu - Tu, z||) + f(||TSx_n - Tx_n, z||)] \\ &= \alpha[f(||TSu - Tu, z||) + f(||Tx_{n+1} - Tx_n, z||)]. \end{aligned}$$

holds true. For $n \to \infty$, the continuous of f and T and the properties of 2-norm imply that

$f(||TSu - Tu, z||) \leq \alpha[f||TSu - Tu, z||) + f(0)]$,

for each $z \in L$, holds true. But, $\alpha \in (0, 1)$ and $f^{-1}(0) = \{0\}$, therefore, the latter inequality implies that $||TSu - Tu, z|| = 0$, for each $z \in L$, that is $TSu = Tu$. Finally, T is injection, therefore $Su = u$, that is it exists a fixed point for S. Let $u, v \in X$ be two fixed points for S, i.e. $Su = u$ and $Sv = v$. Then (3) implies that

$$\begin{aligned} f(||Tu - Tv, z||) &= f(||TSu - TSv, z||) \\ &\leq \alpha[f(||Tu - TSu, z||) + f(||Tv - TSv, z||)] \\ &= 0, \end{aligned}$$

for each $z \in L$, that is $||Tu - Tv, z|| = 0$, for each $z \in L$, i.e. $Tu = Tv$. But, since T is injection, we get that $u = v$, that is T has a unique fixed point. Finally, the arbitrarily of $x_0 \in L$ and the above stated, imply that for each $x_0 \in L$ the sequence $\{S^n x_0\}$ converges to the unique fixed point for S.

Consequence 1. Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$, $f \in \Theta$ and $T : L \to L$ is continuous, injection and sequentially convergent mapping. If there exists $\lambda \in (0, 1)$ such that

$$f(||TSx - TSy, z||) \leq \lambda\sqrt{f(||Tx - TSx, z||)f(||Ty - TSy, z||)}$$

for all $x, y, z \in L$, holds true, then S has a unique fixed point.

Proof. The proof is directly implied by the arithmetic-geometric inequality mean and the Theorem 1, for $\alpha = \frac{\lambda}{2}$.

Consequence 2. Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$, $f \in \Theta$. If there exists $\alpha \in (0, \frac{1}{2})$ such that

$$f(||Sx - Sy, z||) \leq \alpha[f(||x - Sx, z||) + f(||y - Sy, z||)] \qquad (6)$$

, (6) for all $x, y, z \in L$, holds true, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to that fixed point.

Proof. The mapping $Tx = x$, for each $x \in L$ is continuous, injection and sequentially convergent. The statement is directly implied by the Theorem 1 for $Tx = x$.

Consequence 3. ([1]). Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$ and $T : L \to L$ is continuous, injection and sequentially convergent mapping. If there exists $\alpha \in (0, \frac{1}{2})$ such that

$$||TSx - TSy, z|| \leq \alpha(||Tx - TSx, z|| + ||Ty - TSy, z||)$$

for all $x, y, z \in L$, holds true, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to that fixed point.

Proof. The function $f(t) = t$, $t \geq 0$ is monotony increasing and $f^{-1}(0) = \{0\}$. The statement is directly implied by Theorem 1 for $f(t) = t$. □

Remark 1. For $f(t) = t$, $t \geq 0$ (1) is implied by (6), that is Theorem 1, [3] is implied by Consequence 2.

Theorem 2. Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$, $f \in \Theta$ and $T : L \to L$ is continuous, injection and sequentially convergent mapping. If there exists $\alpha \in (0, \frac{1}{2})$ such that

$$f(||TSx - TSy, z||^2) \leq \alpha[f(||Tx - TSx, z||^2) + f(||Ty - TSy, z||^2)] \qquad (7)$$

for all $x, y, z \in L$, holds true, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to that fixed point.

Proof. Let x_0 be any point in L and the sequence $\{x_n\}$ be defined as the following $x_{n+1} = Sx_n$, $n = 0, 1, 2, 3, ...$ The inequality (7), analogously as the Proof of Theorem 1, implies that

$$f(||Tx_{n+1} - Tx_n, z||^2) \ \leq \lambda f(||Tx_n - Tx_{n-1}, z||^2) \qquad (8)$$

for each $n = 0, 1, 2, 3, ...$ and $z \in L$, for $\lambda = \frac{\alpha}{1-\alpha} < 1$, holds true. Further, by applying (8) analogously as the proof of Theorem 1, the sequence $\{x_n\}$ is convergent, i.e. it exists $u \in L$ such that $\lim_{n\to\infty} x_n = u$. Since T is continuous, we get that $\lim_{n\to\infty} Tx_n = Tu$. The inequality (7), analogously as the proof of Theorem 1, we get that

$f(||TSu - Tx_{n+1}, z||^2) \ \leq \alpha[f(||TSu - Tu, z||^2) + f(||Tx_{n+1} - Tx_n, z||^2)].$,

holds true. For $n \to \infty$, the latter is transformed as the following

$f(||TSu - Tu, z||^2) \leq \alpha[f||TSu - Tu, z||^2) + f(0)]$, for each $z \in L$. Analogously, as the ?f Theorem 1 we get that $Su = u$, that is S has a fixed point. Let $u, v \in X$ be fixed points for i.e. $Su = u$ and $Sv = v$. Then, (7) implies

$$\begin{aligned} f(||Tu - Tv, z||^2) &= f(||TSu - TSv, z||^2) \\ &\leq \alpha[f(||Tu - TSu, z||^2) + f(||Tv - TSv, z||^2)] \\ &= 0, \end{aligned}$$

for each $z \in L$. Therefore, we get that $u = v$, i.e. S has a unique fixed point. Finally, the arbitrarily of $x_0 \in L$ and the above stated, imply that for each $x_0 \in L$ the sequence $\{S^n x_0\}$ converges to the unique fixed point for S. □ **Consequence 4.** Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$ and $f \in \Theta$. If there exists $\alpha \in (0, \frac{1}{2})$ such that

$$f(||Sx - Sy, z||^2) \leq \alpha[f(||x - Sx, z||^2) + f(||y - Sy, z||^2)] \tag{9}$$

for all $x, y, z \in L$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to that fixed point.

Proof. For $Tx = x$ in Theorem 2, we get the given statement.□

Consequence 5. ([1]). Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$ and a mapping $T : L \to L$ is such that it is continuous, injection and sequentially convergent mapping. If there exists $\alpha \in (0, \frac{1}{2})$ such tahat $||TSx - TSy, z||^2 \leq \alpha(||Tx - TSx, z||^2 + ||Ty - TSy, z||^2)$, for all $x, y, z \in L$, holds true, then there is a unique fixed point for S and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point. *Proof.* For $f(t) = t$ in Theorem 2, we get the given statement. □

Remark 2. For $f(t) = t$, $t \geq 0$ the condition (9) in Consequence 4 is transformed as the following

$||Sx - Sy, z||^2 \leq \alpha[||x - Sx, z||^2 + ||y - Sy, z||^2]$,

that is Consequence 7, [1] is implied by Consequence 4.

Theorem 3. Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$ and a mapping $T : L \to L$ be such that it is continuous, injection and sequentially convergent mapping and $f \in \Theta$ be such that $f(a + b) \leq f(a) + f(b)$, for all $a, b \geq 0$. If there exists $\alpha \in (0, \frac{1}{2})$ such that

$$f(||TSx - TSy, z||) \leq \alpha[f(||Tx - TSy, z||) + f(||Ty - TSx, z||)] \tag{10}$$

holds true, then there is a unique fixed point for S and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. Let x_0 be any point in L and the sequence $\{x_n\}$ be defined as the following $x_{n+1} = Sx_n$, $n = 0, 1, 2, 3, ...$ The inequality (10) and the properties of f imply

$$\begin{aligned} f(||Tx_{n+1} - Tx_n, z||) &= f(||TSx_n - TSx_{n-1}, z||) \\ &\leq \alpha[f(||Tx_n - TSx_{n-1}, z||) + f(||Tx_{n-1} - TSx_n, z||)] \end{aligned}$$

$$= \alpha f(||Tx_{n-1} - Tx_{n+1}, z||)$$
$$\leq \alpha f(||Tx_{n-1} - Tx_n, z|| + ||Tx_n - Tx_{n+1}, z||)$$
$$\leq \alpha[f(||Tx_{n-1} - Tx_n, z||) + f(||Tx_n - Tx_{n+1}, z||)]$$

So, $f(||Tx_{n+1} - Tx_n, z||) \leq \lambda f(||Tx_n - Tx_{n-1}, z||)$, for each $n = 0, 1, 2, 3, ...$ and each $z \in L$, for $\lambda = \frac{\alpha}{1-\alpha} < 1$. Analogously, as the proof of Theorem 1, we conclude that the sequence $\{x_n\}$ is convergent, i.e. it exists $u \in L$ such that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} Tx_n = Tu$. Further, for each $z \in L$

$$\begin{aligned} f(||TSu - Tx_{n+1}, z||) &= f(||TSu - TSx_n, z||) \\ &\leq \alpha[f(||Tu - TSx_n, z||) + f(||Tx_n - TSu, z||)] \\ &= \alpha[f(||Tu - Tx_{n+1}, z||) + f(||Tx_n - TSu, z||)]. \end{aligned}$$

holds true. For $n \to \infty$, the continuous of f and T and the properties of 2-norm imply that $f(||TSu - Tu, z||) \leq \alpha[f||TSu - Tu, z||) + f(0)]$, for each $z \in L$, holds true. Now, analogously as the proof of Theorem 1, we get that $Su = u$, that is it exists a unique fixed point for S. Let $u, v \in X$ be two fixed point for S, i.e. $Su = u$ and $Sv = v$. Then (3) implies that the following

$$\begin{aligned} f(||Tu - Tv, z||) &= f(||TSu - TSv, z||) \\ &\leq \alpha[f(||Tu - TSv, z||) + f(||Tv - TSu, z||)] \\ &= 2\alpha f(||Tu - Tv, z||), \end{aligned}$$

for each $z \in L$, holds true. And since $2\alpha < 1$ we get that $||Tu - Tv, z|| = 0$, for each? $z \in L$, holds true. The latter implies that $u = v$, that is there exists a unique fix point for S. □

Consequence 6. Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$, the mapping $T : L \to L$ be such that it is continuous, injection and sequentially convergent mapping and function $f \in \Theta$ be such that $f(a+b) \leq f(a)+f(b)$, for all $a, b \geq 0$. If there exists $\lambda \in (0, 1)$ so that

$f(||TSx - TSy, z||) \leq \lambda\sqrt{f(||Tx - TSy, z||)f(||Ty - TSx, z||)}$,

for all $x, y, z \in L$, holds true, then there is a unique fixed point for S.

Proof. The proof is directly implied by the arithmetic-geometric mean inequality and Theorem 3, for $\alpha = \frac{\lambda}{2}$. □

Consequence 7. Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$ and $f \in \Theta$ be such that $f(a + b) \leq f(a) + f(b)$, for all $a, b \geq 0$. If there exists $\alpha \in (0, \frac{1}{2})$ so that

$$f(||Sx - Sy, z||) \leq \alpha[f(||x - Sy, z||) + f(||y - Sx, z||)] \qquad (6)$$

, for all $x, y, z \in L$, holds true , then there is a unique fixed point for S and for each $x_0 \in X$ the sequence $\{S^n x_0\}$converges to that fixed point.

Proof. For $Tx = x$ in Theorem 3, we get the given statement. □

bf Consequence 8. ([1]). Let $(L, ||\cdot,\cdot||)$ be a 2-Banach space, $S : L \to L$ and the mapping $T : L \to L$ be such that it is continuous, injection and sequentially convergent. If there exists $\alpha \in (0, \frac{1}{2})$ so that $||TSx - TSy, z|| \leq \alpha(||Tx - TSy, z|| + ||Ty - TSx, z||)$, for all $x, y, z \in L$, holds true, then there is a unique fixed point for S and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to that fixed point.

Proof. For $f(t) = t$ in Theorem 2, we get the given statement.□

Remark 3. For $f(t) = t$, $t \geq 0$ (1) is implied by (6), that is Theorem 1, [3] is implied by Consequence 2.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHORS CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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