Prove that in any triangle ABC,

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge\sqrt{\frac{6R}{r}}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru , Solution 2 by Adil Abdullayev – Baku – Azerbaidjian , Solution 3 by Martin Lukarevski – Skopje – Macedonia , Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia Solution 5 by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge\sqrt{\frac{6R}{r}}$$

Recordar lo siguiente:

$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{4(p-a)(p-b)(p-c)}{abc},$$

$$\frac{2S}{bc}=\sin A,\frac{2S}{ac}=\sin B,\frac{2S}{ab}=\sin C$$

$$S=\sqrt{p(p-a)(p-b)(p-c)},p=\frac{a+b+c}{2},ab+bc+ac\ge\sqrt{3abc(a+b+c)}$$

Reemplazando en la desigualdad:

$$\frac{bc}{2S}+\frac{ac}{2S}+\frac{ab}{2S}\ge\sqrt{\frac{6}{4}\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}}\to\sqrt{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left(\frac{bc}{2S}+\frac{ac}{2S}+\frac{ab}{2S}\right)\ge\frac{\sqrt{6}}{2}$$

$$\Rightarrow \sqrt{\frac{(p-a)(p-b)(p-c)}{abc}}\left(\sum \frac{bc}{2\sqrt{p(p-a)(p-b)(p-c)}}\right) \geq \frac{\sqrt{6}}{2}$$

$$\Rightarrow \frac{bc}{\sqrt{abcp}} + \frac{ac}{\sqrt{abcp}} + \frac{ab}{\sqrt{abcp}} = \frac{\sqrt{2}}{\sqrt{abc}\sqrt{(a+b+c)}}(bc+ac+ab) \geq$$

$$\geq \frac{\sqrt{2}\sqrt{3abc(a+b+c)}}{\sqrt{abc(a+b+c)}} = \sqrt{6} \ \textit{(LQQD)}$$

Solution 2 by Adil Abdullayev – Baku – Azerbaidjian

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} = \frac{2R}{a} + \frac{2R}{b} + \frac{2R}{c} =$$

$$= 2R \cdot \frac{ab+bc+ca}{abc} = \frac{ab+bc+ca}{2rp} \geq \sqrt{\frac{6R}{r}} \leftrightarrow$$

$$(ab+bc+ca)^2 \geq (2rp)^2 \cdot \frac{6R}{r} =$$

$$= (a+b+c)^2 \cdot 6Rr = (a+b+c)^2 \cdot 6 \cdot \frac{abc}{4S} \cdot \frac{2S}{a+b+c} =$$

$$= 3abc(a+b+c)^2 \leftrightarrow (ab+bc+ca)^2 \geq 3abc(a+b+c) \ldots \textit{(A)}$$

$$\textit{Let } ab = x; bc = y; ca = z$$

$$\textit{(A)} \leftrightarrow (x+y+z)^2 \geq 3(xy+yz+zx)$$

Solution 3 by Martin Lukarevski – Skopje – Macedonia

$$\sum \frac{1}{\sin A} = 2R\sum \frac{1}{a} \geq 2R\sqrt{3\sum \frac{1}{ab}} = \sqrt{\frac{6R}{r}}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \geq \sqrt{\frac{6R}{2}}$$

$$ab+bc+ca\geq\sqrt{3abc(a+b+c)}|:2S$$

$$\frac{ab}{2S}+\frac{bc}{2S}+\frac{ca}{2S}\geq\sqrt{\frac{3abc(a+b+c)}{4S^2}}$$

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\geq\sqrt{\frac{6abc\cdot p}{4\cdot\frac{(abc)^2}{16R^2}}}=\sqrt{\frac{6\cdot R\cdot p}{\frac{abc}{4R}}}=\sqrt{\frac{6R\cdot p}{S}}=\sqrt{\frac{6R}{2}}$$

Solution 5 by George Apostolopoulos – Messolonghi – Greece

We have (Law of Sines)

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}=\frac{2R}{a}+\frac{2R}{b}+\frac{2R}{c}=2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

It is well-known that

$$\frac{1}{4r^2}\geq\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{1}{2Rr}$$

So $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{3}{2Rr}}$

Now

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}=2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 2R\sqrt{\frac{3}{2Rr}}=\sqrt{\frac{4R^2\cdot 3}{2Rr}}=\sqrt{\frac{6R}{r}}$$

Equality holds when the triangle ABC is equilateral.