In $\Delta\, ABC$:

SZOLLOSY'S INEQUALITY

$$\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\le 3R\sqrt{s}$$

Proof 1 by George Apostolopoulos-Messolonghi-Greece, Proof 2 by Serban George Florin-Romania , Proof 3 by Soumitra Mandal-Chandar Nagore-India Proof 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Proof 5 by Martin Lukarevski-Stip-Macedonia

Proof 1 by George Apostolopoulos-Messolonghi-Greece

From Cauchy – Schwarz Inequality, we have

$$\left(\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\right)^2\le$$

$$\le(bc+ca+ab)(s-a+s-b+s-c)\le$$

$$(a^2+b^2+c^2)\cdot s\le 9R^2\cdot s$$

Namely:

$$\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\le 3R\cdot\sqrt{s}$$

Equality holds when the triangle is equilateral.

Proof 2 by Serban George Florin-Romania

$$\sum\sqrt{bc(s-a)}\le 3R\sqrt{s},\ s=\frac{a+b+c}{2}$$

$$\left(\sum\sqrt{bc(s-a)}\right)^2=\left(\sum\sqrt{bc}\cdot\sqrt{s-a}\right)^2\overset{CBS}{\le}\left(\sum\sqrt{bc}^{\,2}\right)\left(\sum\sqrt{s-a}^{\,2}\right)=$$

$$=(ab+bc+ac)(3s-2s)=s(ab+bc+ac)$$

$$\Rightarrow\sum\sqrt{bc(s-a)}\le\sqrt{s}\cdot\sqrt{ab+bc+ac}\le 3R\cdot\sqrt{s}$$

$$\Rightarrow \sqrt{ab + bc + ac} \leq 3R \Rightarrow ab + bc + ac = R^2$$

$$ab + bc + ac \leq a^2 + b^2 + c^2 = 2p^2 - 2r^2 - 8Rr \leq 9r^2$$

$$\Rightarrow 2p^2 \leq 9R^2 + 2r^2 + 8Rr$$

$$\textit{GERRETSEN}\ p^2 \leq 4R^2 + 4Rr + 3r^2 | \cdot 2$$

$$2p^2 \leq 8R^2 + 8Rr + 6r^2 \leq 9R^2 + 2r^2 + 8Rr$$

$$\Rightarrow R^2 \geq 4r^2 \Rightarrow R \geq 2r \ \textit{(Euler)}$$

Proof 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} \sqrt{bc(s-a)} \overset{Cauchy-Schwarz}{\leq} \sqrt{(ab + bc + ca)(s - a + s - b + s - c)}$$

$$= \sqrt{s(ab + bc + ca)} \leq 3R\sqrt{s}\ [\because ab + bc + ca \leq 9R^2]$$

Proof 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$LHS \leq RHS | \cdot 2\sqrt{s}$$

$$\sum_{\Delta} 2 \cdot \sqrt{bc \cdot s(s-a)} \leq 6R \cdot s \ \textit{(ASSURE)}$$

$$\sum_{\Delta} (b + c) \cdot \frac{2 \cdot \sqrt{bc \cdot s(s-a)}}{b + c} = \sum_{\Delta} (b + c) \cdot l_a =$$

$$\overset{CBS}{\leq} \sqrt{\sum_{\Delta} (b + c)^2 \cdot \sum_{\Delta} l_a^2} \quad (*)$$

$$\textbf{a)} \sum (b + c)^2 = 2 \cdot \left(\sum (a^2 + bc)\right) \leq 4 \cdot \sum a^2 \leq 36R^2$$

$$\textbf{b)} \sum l_a^2 = \sum \left(\sqrt{s(s-a)}\right)^2 = s(s - a + s - b + s - c) = s^2$$

(*); a); b) $\Rightarrow$

$$\sum (b + c) \cdot l_a \leq \sqrt{\sum_{\Delta} (b + c)^2 \cdot \sum_{\Delta} l_a^2} \leq \sqrt{36R^2 \cdot s^2} = 6R \cdot s$$

Szollosy's inequality. In a triangle ABC let a, b, c be the sides of the triangle, s its semiperimeter and R the circumradius. Prove that

$$\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\leqslant 3R\sqrt{s}.$$

Solution. The stronger inequality

$$\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\leqslant 2(R+r)\sqrt{s}$$

holds. We use an equivalent form of the Gerretsen's inequality [**1**]

$$ab+bc+ca\leqslant 4(R+r)^2.$$

By Cauchy-Schwarz, we get

$$\begin{aligned}\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)} &\leqslant (ab+bc+ca)^{1/2}\left((s-a)+(s-b)+(s-c)\right)^{1/2}\\ &\leqslant 2(R+r)\sqrt{s}\end{aligned}$$

which proves the inequality.

References

[1] M. Lukarevski, *An alternate proof of Gerretsen's inequalities*, Elem. Math. **72**, (2017), 2-8

Martin Lukarevski, Department of Mathematics and Statistics, University "Goce Delcev" - Stip, Macedonia

E-mail address: `martin.lukarevski@ugd.edu.mk`