Prove that in all triangles ABC holds the inequality

$$\sum_{cyc}\frac{1}{(\sin A+\sin B)^2}\geq 1.$$

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Solutions 1,2 by Kevin Soto Palacios – Huarmey – Peru , Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solutions 4,5 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 6 by Soumitra Mandal-Chandar Nagore-India Solution 7 by Soumava Chakraborty-Kolkata-India, Solution 8 by Martin Lukarevski-Stip-Macedonia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC, la siguiente desigualdad

$$\sum\frac{1}{(\sin A+\sin B)^2}\geq 1\quad \textbf{(A)}$$

Siendo x,y,z números R^+, se cumple la siguiente desigualdad:

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}$$

Realizamos las siguientes sustituciones

$$x=\sin A>0,\ y=\sin B>0,\ z=\sin C>0$$

Luego $\forall\ x,y,z\in R$ se cumple lo siguiente:

$$xy+yz+zx\leq x^2+y^2+z^2=\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C\leq\frac{9}{4}$$

Finalmente la desigualdad es equivalente

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\geq\frac{9}{4(xy+yz+zx)}\geq 1\quad \textbf{(LQQD)}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC, la siguiente desigualdad

$$\sum\frac{1}{(\sin A+\sin B)^2}\geq 1 \quad \textbf{(A)}$$

En un triángulo ABC $\rightarrow \sin A, \sin B, \sin C > 0$

Denotemos los siguientes cambios de variables

$$x=\sin A+\sin B>0, y=\sin B+\sin C>0, z=\sin C+\sin A>0$$

Cumpliéndose que:

$$x+y+z\leq 3\sqrt{3}\wedge(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\sqrt{3}$$

Por la desigualdad de Cauchy en (A)

$$\sum\frac{1}{x^2}\geq\frac{1}{3}\left(\sum\frac{1}{x}\right)^2\geq\frac{1}{3}\left(\frac{9}{x+y+z}\right)^2\geq\frac{1}{3}\cdot 3=1 \quad \textbf{(LQQD)}$$

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\sum\frac{1}{(\sin a+\sin b)^2}\geq\sum\frac{1}{2(\sin^2 a+\sin^2 b)}$$

$$a=2R\cdot\sin a, \qquad b=2R\cdot\sin b, \quad c=2R\cdot\sin c$$

$$\frac{2R^2}{a^2+b^2}+\frac{2R^2}{a^2+c^2}+\frac{2R^2}{b^2+c^2}\overset{C-B-C}{\geq}\frac{2(R+R+R)^2}{2(a^2+b^2+c^2)}=\frac{9R^2}{a^2+b^2+c^2}\geq 1$$

because $9R^2\geq a^2+b^2+c^2$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum_{cyc}\frac{1}{(\sin A+\sin B)^2}=\sum_{cyc}\frac{1^3}{(\sin A+\sin B)^2}\geq$$

$$\geq \frac{(1+1+1)^3}{\left(2\cdot(\sin A+\sin B+\sin C)\right)^2}=\frac{27}{(2\cdot\sum\sin A)^2}\geq\frac{27}{\left(2\cdot\frac{3\sqrt{3}}{2}\right)^2}=1$$

Solution 5 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sin A=\frac{a}{2R}\dots$$

$$\sum_{cyc}\frac{4R^2}{(a+b)^2}=4R^2\sum_{cyc}\frac{1}{a^2+2ab+b^2}=$$

$$=4R^2\cdot\sum_{cyc}\frac{1^2}{a^2+2ab+b^2}\geq 4R^2\cdot\left(\frac{(1+1+1)^2}{2(a^2+b^2+c^2+ab+bc+ca)}\right)$$

$$=18R^2\cdot\frac{1}{a^2+b^2+c^2+ab+bc+ca}\geq 1\quad \textit{(ASSURE)}$$

$$18R^2\geq a^2+b^2+c^2+ab+bc+ca$$

$$a^2+b^2+c^2=2p^2-8Rr-2r^2$$

$$ab+bc+ca=p^2+4Rr+r^2$$

$$p^2\leq 4R^2+4Rr+3r^2$$

$$18R^2\geq 3p^2-4Rr-r^2$$

$$3p^2\leq 18R^2+4Rr+r^2$$

$$3p^2\leq 3(4R^2+4Rr+3r^2)\leq 18R^2+4Rr+r^2$$

$$6R^2-8Rr-8r^2\geq 0$$

$$3R^2-4Rr-4r^2\geq 0$$

$$3R^2=2R^2+r^2\geq 4Rr+4r^2\textit{(Euler)}$$

Solution 6 by Soumitra Mandal-Chandar Nagore-India

We know, $\frac{3}{2}\geq\sin A+\sin B+\sin C$, $\sin A=\frac{a}{2R}$, $\sin B=\frac{b}{2R}$ ***and*** $\sin C=\frac{c}{2R}$

Where a, b, c ***are the sides of a triangle and*** R = ***circum – radius***

$$9R^2 \geq a^2 + b^2 + c^2 \Rightarrow 9R^2 \geq \frac{(a+b+c)^2}{3} \Rightarrow 27R^2 \geq (a+b+c)^2$$

$$\therefore \sum_{cyc} \frac{1}{(\sin A + \sin B)^2} = 4R^2 \sum_{cyc} \frac{1}{(a+b)^2}$$

$$\geq 4R^2 \cdot \frac{27}{(a+b+b+c+c+a)^2} \left[\because \frac{1}{3}\sum_{cyc} \frac{1}{(a+b)^2} \geq \frac{9}{4} \cdot \frac{1}{(a+b+c)^2}\right]$$

$= \frac{27R^2}{(a+b+c)^2} \geq 1$ ***(proved) equality at*** $A = B = C = \frac{\pi}{3}$

Solution 7 by Soumava Chakraborty-Kolkata-India

$$\sin A + \sin B = 2\cos\frac{C}{2}\cos\frac{A-B}{2}$$

$$\because -\frac{\pi}{2} < \frac{A-B}{2} < \frac{\pi}{2},$$

$$\therefore 0 < \cos\frac{A-B}{2} \leq 1$$

$$\Rightarrow 2\cos\frac{C}{2}\cos\frac{A-B}{2} \leq 2\cos\frac{C}{2}$$

$$\Rightarrow \sin A + \sin B \leq 2\cos\frac{C}{2} \Rightarrow (\sin A + \sin B)^2 \overset{(a)}{\leq} 4\cos^2\frac{C}{2}$$

Similarly, $(\sin B + \sin C)^2 \overset{(b)}{\leq} 4\cos^2\frac{A}{2}$ ***and*** $(\sin C + \sin A)^2 \overset{(c)}{\leq} 4\sin^2\frac{B}{2}$

Using (a), (b), (c) $\sum \frac{1}{(\sin A + \sin B)^2} \geq \frac{1}{4}\sum \sec^2\frac{A}{2}$

Let $f(x) = \sec^2\left(\frac{x}{2}\right)\ \forall x \in (0, \pi)$

$$f''(x) = \sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right) + \frac{1}{2}\sec^4\left(\frac{x}{2}\right) > 0$$

$$\therefore f(x) \text{ is convex on } (0,\pi)$$

$$\therefore \sum \frac{1}{(\sin A + \sin B)^2} \geq \frac{1}{4}\sum \sec^2\frac{A}{2} \overset{Jensen}{\geq} \frac{1}{4}\cdot 3\sec^2\left(\frac{A+B+C}{6}\right)$$

$$= \frac{1}{4}\cdot 3\cdot\left(\frac{2}{\sqrt{3}}\right)^2 = 1 \quad \textit{(Proved)}$$

Solution 8 by Martin Lukarevski-Stip-Macedonia

Prove that in all triangles ABC holds the inequality

$$\sum \frac{1}{(\sin A + \sin B)^2} \geqslant 1. \tag{1}$$

Solution. We use the well-known inequality $a^2+b^2+c^2 \leqslant 9R^2$. Since $ab+bc+ca \leqslant a^2+b^2+c^2$, the inequality

$$\sum (a+b)^2 \leqslant 36R^2$$

holds. By the law of sines the inequality (1) is equivalent to

$$\sum \frac{1}{(a+b)^2} \geqslant \frac{1}{4R^2}.$$

By Cauchy-Schwarz

$$\sum \frac{1}{(a+b)^2} \geqslant \frac{9}{\sum (a+b)^2} \geqslant \frac{1}{4R^2},$$

and we are done.

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