Wavelets and Continuous Wavelet Transform

Jasmina Veta Buralieva
University Goce Delcev / Faculty of Computer Sciences, Stip, Republic of Macedonia
jasmina.buralieva@ugd.edu.mk,

Abstract - The concepts of wavelet theory were provided by Meyer, Mallat, Daubechies and many others.Wavelets are well localized, oscillatory functions which provide a basis of $L^2(\mathbb{R})$ and can be modified to a basis of $L^2([a,b])$, where $[a,b]$ is a bounded domain. Wavelet transform or wavelet analysis is a recently developed mathematical tool for signal analysis. In this paper are shown some relation for the wavelet transform and using Wolfram Mathematica 10 is given the wavelet transform of the signal $f(x)=\sin x$. A prove that multiplication of real number with wavelet and sum of two wavelets are wavelets is also provided.

I. INTRODUCTION

Wavelets have been around since the late 1980s, and have found many applications in signal processing, numerical analysis, operator theory, and other fields [2, 5, 10]. In 1982, the French geophysicist Jean Morlet introduced the concept of a "wavelet", which means a small wave and studied wavelet transform as a new tool for seismic signal analysis [1, 12]. The wavelet transform is a tool that cuts up data or functions or operators into different frequency components, and then studies each component with a resolution matched to its scale. The concepts of wavelet theory were provided by Meyer, Mallat, Daubechies and many others, [3,4,6,7,8,12,15]. Wavelets are well localized, oscillatory functions which provide a basis of $L^2(\mathbb{R})$ and can be modified to a basis of $L^2([a,b])$ where $[a,b]$ is a bounded domain [8,9]. Their well localization allows local variations of the problem to be analyzed at various levels of resolution.

Wavelets have generated significant interest from both theoretical and applied researchers over the last few decades. In areas such as time-series analysis, approximation theory and numerical solutions of differential equations, wavelets are recognized as powerful weapons not just tools. In [13] is found the approximate solution in a different levels of the Laplace partial differential equation of third order using the Wavelet-Galerkin method. But in [11, 14] the same method is used to solve differential equation for a different wavelets.

The outline of this paper is as follows: In Section II we describe the spaces of functions and we define wavelets, translation and modulation of a function and wavelet transform. In Section III are shown some relations for the wavelet transform and using Wolfram Mathematica 10 is given the wavelet transform of the signal $f(x)=\sin x$. A prove that multiplication of real number with wavelet and sum of two wavelets are wavelets is also provided.

II. WAVELETS AND CONTINUOUS WAVELET TRANSFORM

A. Spaces of functions

$L^2(\mathbb{R})$ is a Hilbert space of square integrable functions on the real line with the inner product

$$\langle f,g\rangle=\int_{\mathbb{R}} f(t)\bar{g}(t)dt \tag{1}$$

where $\bar{g}(t)$ is a complex conjugate of $g(t)$. The Fourier transform of a function $f\in L^2(\mathbb{R})$ is given with

$$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

$C^2([a,b])$ is the space of functions on $[a,b]$, $\forall\, a,b\in R$ with continuous derivatives up to order 2. The convolution of two integrable functions i.e. $f,g\in L^1(R)$ is defined as

$$(f*g)(x)=\int_R f(y)g(x-y)dy\,\cdot$$

B. Wavelets

A function $\psi\in L^2(\mathbb{R})$ is called a wavelet if it has zero average on $(-\infty,+\infty)$ i.e.

$$\int_{-\infty}^{+\infty}\psi(t)dt=0\,\cdot \tag{2}$$

The function $\psi\in L^2(\mathbb{R})$ (called a *wavelet* or *mother wavelet*) is assumed to satisfy the admissibility condition

$$C_\psi=\int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega<\infty, \tag{3}$$

which implies that $\hat{\psi}(0) = \int_{-\infty}^{\infty}\psi(t)dt = 0$.

One can prove that, if $\int_{-\infty}^{\infty}\psi(t)dt = 0$ and $\int_{-\infty}^{\infty}(1+|t|^{\alpha})|\psi(t)|dt < \infty$ for some $\alpha > 0$, then $C_{\psi} < \infty$, [9].

Remark1. Let us note that there is no clasical definition for wavelets. Somethimes functions that satisfies the admisibilitty condition (3) are called wavelets, but usually functions that satisfies the condition that gives the first vanishing moment (2) are called wavelets.

In most situations, it is usefully to restrict ψ to be well localized both in time and frequency domains. For time localization, $\psi(t)$ and its derivatives must decay very rapidly, while for frequency localization, $\hat{\psi}(\omega)$ must decay sufficiently fast as $|\omega| \to \infty$ and $\hat{\psi}(\omega)$ must become a flat in the neighborhood of 0. The flatness is associated with the number of vanishing moments of $\psi(t)$ since $\int_{-\infty}^{\infty}t^k\psi(t)dt = 0 \Leftrightarrow \hat{\psi}^{(k)}(0) = 0$ for k = 0, 1, …, n. It means that larger number of vanishing moments more is the flatness ω is small.

Lemma 1. ([1], Lemma 10.2.1) Let $\varphi(t)$ be a nonzero n-times ($n \geq 1$) differentialble functions such that $\varphi^{(n)}(x) \in L^2(R)$. Then $\psi(x) = \varphi^{(n)}(x)$ is a wavelet.

Corollary 2. ([1], Corollary 10.2.1) For every non-zero element $\psi(x) \in L^2(R)$ with compact support, the following statements are equivalent:

1. The function $\psi(t)$ is a wavelet
2. The condition (3) is satisfied.

Theorem 3. ([1], Theorem 10.2.1) Let ψ be a wavelet and φ a bounded integrable function then the convolution $\psi * \varphi$ is a wavelet.

Example1. (Haar Wavelet) Let

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \dfrac{1}{2} \\ -1, & \dfrac{1}{2} \leq x \leq 1 \\ 0, & x < 0 \text{ or } x > 1 \end{cases} . \tag{4}$$

It is clear that $\psi(x)$ has compact support [0,1] and $\int_R \psi(x)dx = \int_0^{1/2} dx - \int_{1/2}^{1} dx = 0$.

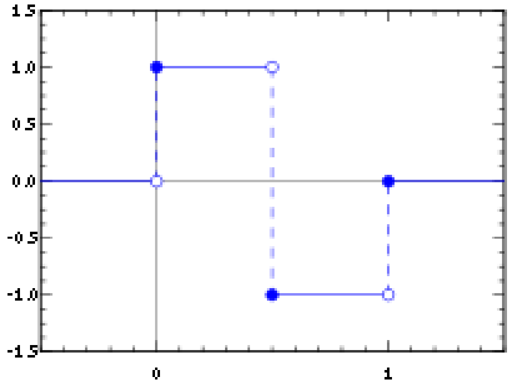


Figure 1. Haar Wavelet

The Fourier transform is

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_o^{1/2} e^{-i\omega x}dx - \frac{1}{\sqrt{2\pi}}\int_{1/2}^{1} e^{-i\omega x}dx$$

$$= \frac{1}{i\omega\sqrt{2\pi}}\left(-2e^{-i\omega/2} + 1 + e^{-i\omega}\right)$$

$$= \frac{1}{i\omega\sqrt{2\pi}}\left(1 - e^{-i\omega/2}\right)^2$$

$$= \frac{1}{i\omega\sqrt{2\pi}}\left(\frac{e^{i\omega/2} - 1}{e^{i\omega/2}}\right)^2 \cdot \frac{1}{\left(2ie^{i\omega/4}\right)^2} \cdot \left(2ie^{i\omega/4}\right)^2$$

$$= \frac{1}{i\omega\sqrt{2\pi}}\left(\frac{e^{i\omega/2} - 1}{2ie^{i\omega/4}}\right)^2 \cdot \frac{1}{\left(e^{i\omega/2}\right)^2} \cdot \left(2ie^{i\omega/4}\right)^2$$

$$= -\frac{1}{i\sqrt{2\pi}} \frac{\left(\sin\frac{\omega}{4}\right)^2}{\frac{\omega}{4}} e^{-\frac{i\omega}{2}} \frac{e^{\frac{i\pi}{2}}}{e^{\frac{i\pi}{2}}} = \frac{1}{\sqrt{2\pi}} \frac{\left(\sin\frac{\omega}{4}\right)^2}{\frac{\omega}{4}} e^{-\frac{i}{2}(\omega-\pi)}.$$

Function (4) satisfy the condition (2) and also the admissibility condition (3), so by the Corollary 2 is a wavelet (see Figure 1).

Example 2. (Mexican Hat Wavelet) The function defined as

$$\psi(x) = \left(1 - x^2\right)e^{-\frac{x^2}{2}} \tag{5}$$

may be writeen as second derivative of non zero n-times ($n \geq 1$) differentialble function $\varphi(x) = e^{-\frac{x^2}{2}}$ such that $\varphi^{(2)}(x) \in L^2(R)$ i.e.

$$\psi(x) = -\frac{d^2}{dx^2}e^{-\frac{x^2}{2}} = \left(1 - x^2\right)e^{-\frac{x^2}{2}}.$$

According to Lemma 1 the function (5) is a wavelet and even more it is known as the Maxican hat wavelet (Figure 2). From Figure 1 and Figure 2 we can note that the Haar wavelet is discontinuous at $x = 0, \frac{1}{2}, 1$ but the Maxican hat wavelet is a continuous on R.

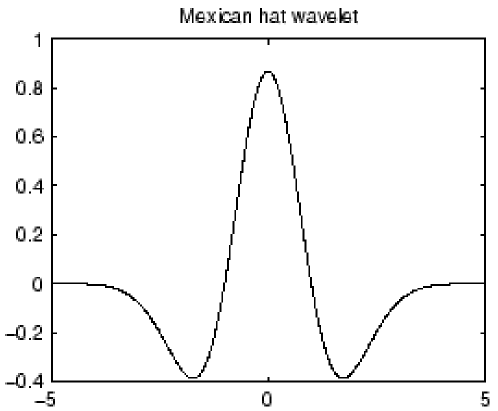


Figure 2. Mexican hat wavelet

Example 3. The convolution of the Haar wavelet and the function $\varphi(x) = e^{-x^2}$, is the function on the Figure 3. According to Theorem 3 this function is a wavelet.

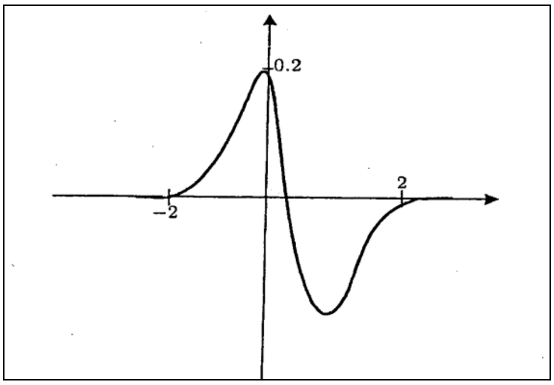


Figure 3. Convolution of the Haar wavelet and $\varphi(x) = e^{-x^2}$

C. *Basic operations*

Let $x \in R$, $a \in R$, $a \neq 0$. With (6) and (7) are given the translation operator and the dilatation operator, respectivelly

$$T_x f(t) = f(t - x) \tag{6}$$

$$D_a f(x) = |a|^{-1/2} f(a^{-1}x). \tag{7}$$

Translation operator T_x retains the form of the function but slip feature along the x-axis depending on the parameter x (see Figure 4). Dilatation operator D_a preserve the form of the function, but it change the scale i.e "width" of the function (see Figure 5 a) and b) by different factor).

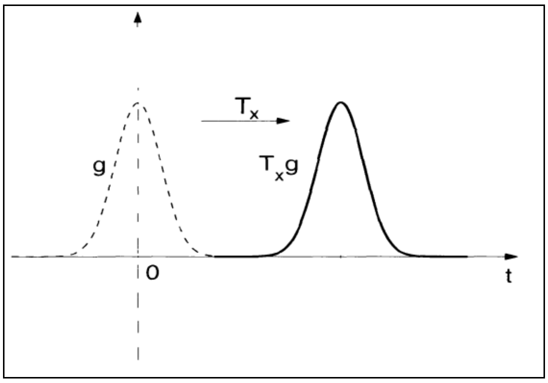
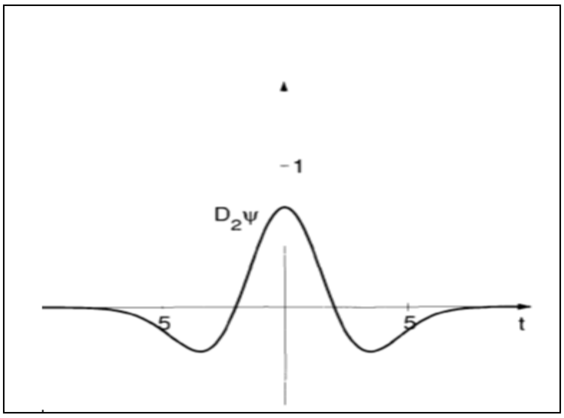
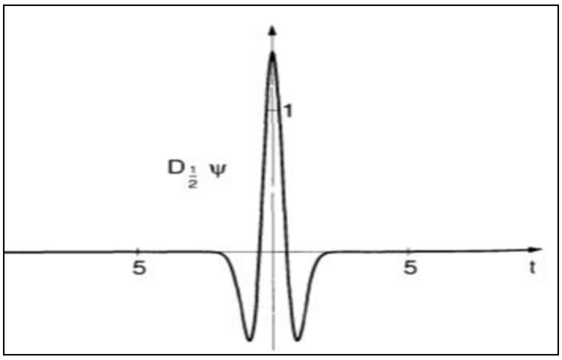


Figure 4. Translated function



a)



b)

Figure 5. Dilatation of the function
a) by factor 2 b) by factor 1/2

D. *The continuous wavelet transform*

The continuous wavelet transform W_ψ of a function $f \in L^2(R)$ with respect to the wavelet ψ is defined as

$$W_\psi f(a,b) = |a|^{-1/2} \int_R f(t)\, \overline{\psi}\left(\frac{t-b}{a}\right) dt \tag{8}$$

where $a \in R \setminus \{0\}$, $b \in R$ and $\overline{\psi}$ denotes complex conjugate. Let ψ is a wavelet (mother wavelet) then

$\{\psi_{a,b}\}$ is a family of functions defined as

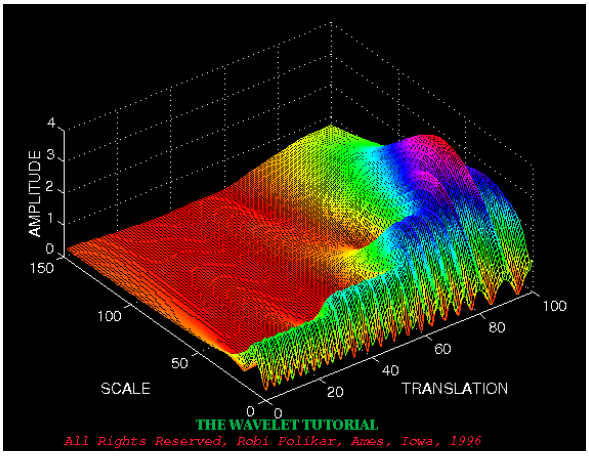
$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right), a \in R\setminus\{0\}, b \in R. \quad (9)$$

Wavelet transform can be written in other forms, such as

$$W_\psi f(a,b) =< f, \psi_{a,b} >$$
$$=< f, T_b D_a \psi >.$$

using (1), (6), (7) and (8). It may be observe that wavelet transform $W_\psi f(b,a)$ is a function of scale or frequency a and time b. $W_\psi f(b,a)$ measures the variation of f in a neighborhood of b. For a compactly supported wavelet, the value of $W_\psi f(b,a)$ depends upon the value of f in a neighborhood of b of size proportional to scale a. If $\operatorname{supp}\psi \subseteq E$ and the set E is centered around the origin then $\operatorname{supp} T_b D_a \psi \subseteq b + aE$ is a neighborhood of b with scale a. On the Figure 7 is given the signal with four frequencies (a) and its wavelet transform (b).

a)



b)

Figure 7. a) Signal with four frequencies
b) Wavelet transform of the signal

III. Main Results

In this section are shown two properties for wavelets, some usefull relations for the wavelet transform and using Wolfram Mathematica 10 is given the wavelet transform of the signal $f(x) = \sin x$.

Problem 1. If $\psi_1(x)$ and $\psi_2(x)$ are two wavelets then for $a \in R$, $a\psi_1(x)$ and $\psi_1(x)+\psi_2(x)$ are wavelets.

Proof:

a) $\psi_1(x)$ and $\psi_2(x)$ are wavelets so they have zero-average on $(-\infty,\infty)$, i.e $\int_{-\infty}^{+\infty}\psi_1(t)dt = 0$, $\int_{-\infty}^{+\infty}\psi_2(t)dt = 0$.

Using this and the additivity of the integral we have

$$\int_{-\infty}^{+\infty}(\psi_1+\psi_2)(t)dt = \int_{-\infty}^{+\infty}\psi_1(t)dt + \int_{-\infty}^{+\infty}\psi_2(t)dt = 0$$

and $\int_{-\infty}^{+\infty}\alpha\psi_1(t)dt = \alpha\int_{-\infty}^{+\infty}\psi_1(t)dt = 0$.

According to (2) we can conclude that $\psi_1(x)+\psi_2(x)$ and $a\psi_1(x)$ are wavelets.

b) If we assume that $\psi_1(x)$ and $\psi_2(x)$ are wavelets with compact support, using the collorary 2 they satisfy the condition (3) so the Fourier transform for $\psi(x) = \psi_1(x)+\psi_2(x)$ is

$$\hat{\psi}(\omega) = \int_{-\infty}^{+\infty}\psi(t)e^{-2\pi i\omega t}dt = \hat{\psi}_1(\omega)+\hat{\psi}_2(\omega)$$

$$C_\psi = \int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega = \int_{-\infty}^{\infty}\frac{|\hat{\psi}_1(\omega)+\hat{\psi}_2(\omega)|^2}{|\omega|}d\omega =$$
$$= \int_{-\infty}^{\infty}\frac{|\hat{\psi}_1(\omega)|^2}{|\omega|}d\omega + \int_{-\infty}^{\infty}\frac{2|\hat{\psi}_1(\omega)||\hat{\psi}_2(\omega)|}{|\omega|}d\omega + \int_{-\infty}^{\infty}\frac{|\hat{\psi}_2(\omega)|^2}{|\omega|}d\omega$$

$$\leq \int_{-\infty}^{\infty}\frac{|\hat{\psi}_1(\omega)|^2}{|\omega|}d\omega + \int_{-\infty}^{\infty}\frac{|\hat{\psi}_2(\omega)|^2}{|\omega|}d\omega +$$
$$+ \int_{-\infty}^{\infty}\frac{2\int_{-\infty}^{\infty}|\psi_1(\omega)e^{-2\pi i\omega t}|dt\int_{-\infty}^{\infty}|\psi_2(\omega)e^{-2\pi i\omega t}|dt}{|\omega|}d\omega < \infty$$

$$C_\psi = \int_{-\infty}^{\infty}\frac{|\alpha\hat{\psi}_1(\omega)|^2}{|\omega|}d\omega = \alpha^2\int_{-\infty}^{\infty}\frac{|\hat{\psi}_1(\omega)|^2}{|\omega|}d\omega < \infty$$

Function $\psi(x)$ and $\alpha\psi_1(x)$ satisfied the admisibillity condition (3), so by collorary 2 $\psi_1(x)+\psi_2(x)$ and $a\psi_1(x)$ are wavelets.

Problem 2. Let ψ and ϕ are wavelets and let $f, g \in L^2(R)$, for $\forall\alpha,\beta \in C$. Prove the relations:

a) $W_\psi\left(\alpha\ f+\beta\ g\right)(a,b)=\alpha W_\psi\ f\ (a,b)+\beta W_\psi\ g\ (a,b)$,

$$W_\psi\,(\alpha f+\beta g)(a,b)=|a|^{-1/2}\int_R(\alpha f+\beta g)(t)\,\overline{\psi}\left(\frac{t-b}{a}\right)dt$$

$$=\alpha|a|^{-1/2}\int_R f\,(t)\,\overline{\psi}\left(\frac{t-b}{a}\right)dt+$$

$$+\beta|a|^{-1/2}\int_R g(t)\,\overline{\psi}\left(\frac{t-b}{a}\right)dt$$

$$=\alpha W_\psi\ f\,(a,b)+\beta W_\psi\,g(a,b)$$

b) $W_\psi\left(T_x\,f\right)(a,b)=W_\psi\ f\ (a,b-x)$, $c\in R$.

$$W_\psi\,(T_x\,f)(a,b)=|a|^{-1/2}\int_R f\,(t-x)\,\overline{\psi}\left(\frac{t-b}{a}\right)dt$$

$$=|a|^{-1/2}\int_R f\,(m)\,\overline{\psi}\left(\frac{m-(-x+b}{a}\right)dm$$

$$=W_\psi\ f\,(a,b-x)$$

c) $W_\psi\left(D_s\,f\right)(a,b)=W_\psi\ f(a/s,b/s)$, $s>0$.

$$W_\psi\,(D_s\,f)(a,b)=|a|^{-1/2}\int_R|s|^{-1/2}\ f\,(s^{-1}t)\,\overline{\psi}\left(\frac{t-b}{a}\right)dt$$

$$=|a|^{-1/2}\int_R f\,(m)\,\overline{\psi}\left(\frac{m-(b/s)}{a/s}\right)dm$$

$$=W_\psi\ f\,(a/s,b/s)$$

d) $W_\psi\ f\,(a,b)=W_f\,\psi(1/a,-b/a),\ a\neq 0$.

$$W_\psi\ f\,(a,b)=|a|^{-1/2}\int_R f\,(t)\,\overline{\psi}\left(\frac{t-b}{a}\right)dt$$

$$=|a|^{-1/2}\int_R\bar{f}\left(\frac{z+(b/a)}{1/a}\right)\psi(z)adz$$

$$=W_f\,\psi(1/a,-b/a)$$

e) $W_{\alpha\psi+\beta\phi}\ f\,(a,b)=\overline{\alpha}\ W_\psi\ f\,(a,b)+\overline{\beta}\,W_\phi\,f\,(a,b)$,

$$W_{\alpha\psi+\beta\varphi}\ f\,(a,b)=|a|^{-1/2}\int_R f\,(t)\,\overline{(\alpha\psi+\beta\varphi)}\left(\frac{t-b}{a}\right)dt$$

$$=|a|^{-1/2}\int_R f\,(t)\,\overline{\alpha\psi}\left(\frac{t-b}{a}\right)dt$$

$$+|a|^{-1/2}\int_R f\,(t)\,\overline{\beta\varphi}\left(\frac{t-b}{a}\right)dt$$

$$=\overline{\alpha}W_\psi\ f\,(a,b)+\overline{\beta}W_\varphi\,f\,(a,b)$$

f) $W_{T_x\psi}\ f\,(a,b)=W_\psi\ f\,(a,b+xa)$, $x\in R$.

$$W_{T_x\psi}\ f\,(a,b)=|a|^{-1/2}\int_R f\,(t)\,\overline{T_x\psi}\left(\frac{t-b}{a}\right)dt$$

$$=|a|^{-1/2}\int_R f\,(t)\overline{\psi}\left(\frac{t-b}{a}-x\right)dt$$

$$=|a|^{-1/2}\int_R f\,(t)\overline{\psi}\left(\frac{t-(b+ax)}{a}\right)dt$$

$$=W_\psi\ f\,(a,b+xa)$$

g) $W_{D_x\psi}\ f\,(a,b)=1/\sqrt{x}\,W_\psi\ f\,(ac,b),\ x>0$.

$$W_{D_x\psi}\ f\,(a,b)=|a|^{-1/2}\int_R f\,(t)\,\overline{D_x\psi}\left(\frac{t-b}{a}\right)dt$$

$$=|a|^{-1/2}\int_R f\,(t)|x|^{-1/2}\,\overline{\psi}\left(x^{-1}\,\frac{t-b}{a}\right)dt$$

$$=|x|^{-1/2}W_\psi\ f\,(ax,b).$$

Problem 3. Compute the wavelet transform of the signal $f\,(x)=\sin\,x$ using the Haar wavelet (4).

Solution:

$$W_\psi\ f\,(a,b)=|a|^{-1/2}\int_R\sin\,x\,\overline{\psi}\,(\frac{x-b}{a})dx=$$

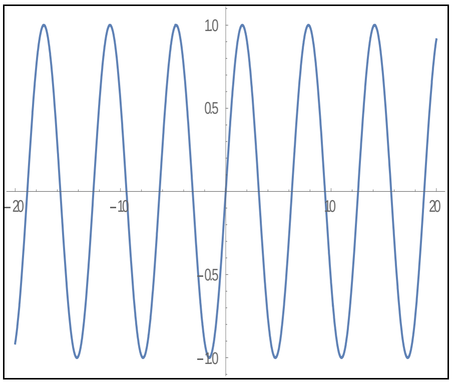
$$=|a|^{-1/2}\left(\int_b^{\frac{a}{2}+b}\sin\,xdx-\int_{\frac{a}{2}+b}^{a+b}\sin\,xdx\right)$$

$$=|a|^{-1/2}\left(\cos b+\cos(a+b)-2\cos\left(\frac{a}{2}+b\right)\right)$$

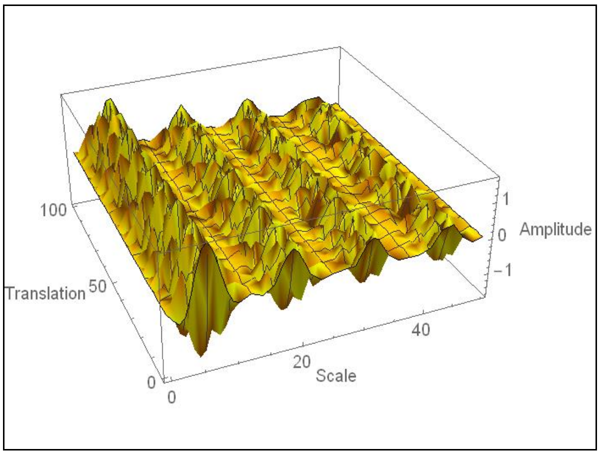
On the Figure 9 is given the signal $f\,(x)=\sin\,x$ and its wavelet transform that we obtain in Wolfram Mathematica 10 using Haar wavelet.

CONCLUSION

From the problem 1 we can conclude that the sum of two wavelets and multiplication of the real number with a wavelet are also wavelets which are very important properties for the wavelet transform. The relations that are proven in problem 2 simplify the calculation of the wavelet transform over the sum of functions, multiplication of the real number and function, dilatated and translated function; or wavelet transform of the function using sum of wavelets, multiplication of the real number and wavelet, dilatated and translated wavelet. In problem 3 we can visually see the wavelet transform of the signal $f\,(x)=\sin\,x$, despite the mathematical calculation.



a)



b)

Figure 8. a) Signal $f(x) = \sin x$, b) Wavelet transform of the signal

REFERENCES

[1] A.H.Siddiqi. Applied Functional Analysis: numerical methods, wavelet methods and image Processing, Markel Dekker, New York, 2004

[2] A.W. Galli, G.T. Heydt, and P.F. Ribeiro, "Exploring the power of wavelet analysis." IEEE Computer Application in Power, Oct 1996,pp.37 – 41.

[3] C. K. Chui. An introduction to wavelets. Academic Press, Inc . Boston, MA. 1992.

[4] D. F. Walnut. An introduction to Wavelet Analysis, Birkhauser, 2004.

[5] E. M. Stein. Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals. Princeton Univ. Press, Princeton, NJ, 1993.With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, **III**

[6] E. Hernandez, G. Weiss. A First Course on Wevelets, CRC Press, Inc., 1996

[7] I. Daubechies. Ten Llectures on Wavelets. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[8] I. Daubechies: Orthonormal bases of compactly suppoted wavelets, Commun. Pure Appl. Math. 41(1988), 909_996.

[9] K. Grochening. Foundations of Time- Frequency Analysis. Applied and Numerical Harmonic Analysis, Birkhauser, Boston, 2001.

[10] K. Gröchenig. Describing functions: atomic decompositions versus frames. Monatsh. Math., 112(1):1-42,1991.

[11] K.Sanja, V.B, Jasmina and H-V. S, Katerina (2013) Wavelet-Galerkin solution of some ordinary differential equations. Proceedings of the XI International Conference ETAI 2013, Ohrid, Republic of Macedonia. ISSN ISBN: 978-9989-630-67-5

[12] S. Mallat. A wavelet tour of signal processing. Academic Press, San Diego, CA, 1998.

[13] V B. Jasmina, H. Elena and H-V S. Katerina, On a numerical solution of the Laplace equation, Advances in Mathematics: Scientific Journal 4 (2015). pp. 209-220. ISSN 1857-8365

[14] V B. Jasmina, K. Sanja and H-V S. Katerina (2014) Wavelet application in solving ordinary differential equations using Galerkin method. Yearbook of the Faculty of Computer Science 2013, 2 (2). pp. 17-26. ISSN 1857- 8691

[15] Y. Meyer. Wavelets. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. 1993. Algorithms & applications. Translated from the French and with a foreword by Robert D. Ryan.