**100.C** (Muhammed Osman Corbali)

Triangle $ABC$ has side-lengths $a = BC$, $b = AC$, $c = AB$ and ex-radii $r_a, r_b, r_c$ where $r_a$ refers to the escribed circle opposite to vertex $A$.

Prove that $\frac{b^2c^2}{r_a^2} + \frac{a^2c^2}{r_b^2} + \frac{a^2b^2}{r_c^2} \geqslant \frac{4}{3}\left(a^2 + b^2 + c^2\right)$.

Several solvers noted that a slightly stronger result holds, namely

$$\frac{b^2c^2}{r_a^2} + \frac{c^2a^2}{r_b^2} + \frac{a^2b^2}{r_c^2} \geqslant 12R^2. \qquad (*)$$

This implies the inequality in **100.C** because of the fact that

$$OH^2 = 9R^2 - \left(a^2 + b^2 + c^2\right) \geqslant 0$$

where $O$ is the circumcentre and $H$ is the orthocentre.

Standard formulae give $r_a = \frac{\Delta}{s - a}$ and $\Delta = \frac{abc}{4R}$ where $\Delta$ is the area of triangle $ABC$. Thus

$$\frac{bc}{r_a} = \frac{4R(s - a)}{a} = 2R\left(\frac{b}{c} + \frac{c}{a} - 1\right)$$

and $$\sum \frac{bc}{r_a} = 2R\left(\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{a} - 3\right) \geqslant 6R$$

by the AM-GM inequality.

Then, by the Cauchy-Schwarz inequality,

$$3\sum \frac{b^2c^2}{r_a^2} \geqslant \left(\sum \frac{bc}{r_a}\right)^2 \geqslant 36R^2$$

which establishes $(*)$. There is equality in both $(*)$ and **100.C** if, and only if, triangle $ABC$ is equilateral.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, M. Lukarevski (2 solutions), J. A. Mundie, P. Nüesch, Z. Retkes (2 solutions), V. Schindler, E. Swylan and the proposer Muhammed Osman Corbali.