SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
ROMANIAN MATHEMATICAL SOCIETY

Filiala Mehedinți - Mehedinți Branch
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In ΔABC:

CIOPLEA'S INEQUALITY – 1

$$\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right)\left(\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\right)\le\frac{9R}{2r}$$

Proof 1 by Kevin Soto Palacios – Huarmey – Peru

Proof 2 by Myagmarsuren Yadamsuren-Ulanbataar-Mongolia

Proof 3 by Soumava Chakraborty – Kolkata – India

Proof 4 by Adil Abdullayev – Baku – Azerbaidian

Proof 5 by Martin Lukarevski-Stip-Macedonia

Proof 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right)\left(\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\right)\le\frac{9R}{2r}$$

1) Tener en cuenta las siguientes desigualdades en un triángulo ABC que han sido demostradas anteriormente:

$$\frac{R}{r}\ge\frac{c}{b}+\frac{b}{c},\ \frac{R}{r}\ge\frac{a}{c}+\frac{c}{a},\ \frac{R}{r}\ge\frac{b}{a}+\frac{a}{b},\ \frac{R}{r}\ge 2$$

$$\Rightarrow 3+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{c}{a}}+\sqrt{\frac{a}{c}}\le\frac{9R}{2r}\ldots\textbf{(A)}$$

Es suficiente probar que:

$$\frac{a}{b}+\frac{b}{a}\ge\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\to\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2-\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)-2=$$

$$=\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}-2\right)\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}+1\right)\geq 0$$

Siendo esto ultimo verdadero ya que: $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\geq 2\rightarrow$

$\rightarrow$ ***(Válido por: MA ≥ MG)***

Por la tanto tenemos en ... (A)

$$\Rightarrow 3+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{c}{a}}+\sqrt{\frac{a}{c}}\leq$$

$$\leq 3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{c}{b}+\frac{b}{c}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)\leq 3+\frac{3R}{r}\leq\frac{3R}{2r}+\frac{6R}{2r}=\frac{9R}{2r}\ \text{... (LQQD)}$$

Proof 2 by Myagmarsuren Yadamsuren-Ulanbataar-Mongolia

$$I=\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right)\left(\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\right)\leq\frac{9R}{2r}$$

$$1\cdot\sqrt{\sin A}+1\cdot\sqrt{\sin B}+1\cdot\sqrt{\sin C}\leq\sqrt{(1^2+1^2+1^2)(\sin A+\sin B+\sin C)}$$

$$1\cdot\sqrt{\sin A}+1\cdot\sqrt{\sin B}+1\cdot\sqrt{\sin C}\leq\sqrt{(1^2+1^2+1^2)\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)}$$

$$I\leq 3\cdot\sqrt{(\sin A+\sin B+\sin C)\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)}$$

$$p=\frac{a+b+c}{2}=\frac{2R\cdot\sin A+2R\cdot\sin B+2R\cdot\sin C}{2}$$

1. $\sin A+\sin B+\sin C=\frac{p}{R}$

2. $\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}=\frac{bc}{\frac{2\cdot bc\cdot\sin A}{2}}+\frac{ac}{\frac{2\cdot ac\cdot\sin B}{2}}+\frac{ab}{\frac{2\cdot ab\cdot\sin C}{2}}=$

$$= \frac{bc}{2S} + \frac{ac}{2S} + \frac{ab}{2S} = \frac{ab + bc + ca}{2S} =$$

$$= \frac{p^2 + 4R \cdot r + r^2}{2 \cdot p \cdot r}$$

$$I \le 3 \cdot \sqrt{\frac{p}{R} \cdot \frac{p^2 + 4R \cdot r + r^2}{2p \cdot r}} = 3 \cdot \sqrt{\frac{p^2 + 4R \cdot r + r^2}{2rR}} \overset{GERRETSEN}{\le}$$

$$\le 3 \cdot \sqrt{\frac{4R^2 + 8R \cdot r + 4r^2}{2R \cdot r}} = 3\sqrt{\frac{2 \cdot (R + r)^2}{R \cdot r}} = 3 \cdot (R + r) \cdot \sqrt{\frac{2}{R \cdot r}} \overset{Euler}{\le}$$

$$\le 3 \cdot \left(R + \frac{R}{2}\right) \cdot \sqrt{\frac{2}{(2r) \cdot r}} = \frac{9R}{2r}$$

Proof 3 by Soumava Chakraborty – Kolkata – India

$$LHS = \frac{1}{\sqrt{2R}}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)\sqrt{2R}\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right) =$$

$$= 3 + \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right) + \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right) + \left(\sqrt{\frac{c}{a}} + \sqrt{\frac{a}{c}}\right) =$$

$$= 3 + \frac{a + b}{\sqrt{ab}} + \frac{b + c}{\sqrt{bc}} + \frac{c + a}{\sqrt{ac}} = 3 + \frac{2s - c}{\sqrt{ab}} + \frac{2s - a}{\sqrt{bc}} + \frac{2s - b}{\sqrt{ac}}$$

$$= 3 + 2s\left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}\right) - \left(\frac{c}{\sqrt{ab}} + \frac{a}{\sqrt{bc}} + \frac{b}{\sqrt{ca}}\right) \quad \textbf{(A)}$$

Now, $\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \overset{CBS}{\le} \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ ***(1)***

Again, $\frac{c}{\sqrt{ab}} + \frac{a}{\sqrt{bc}} + \frac{b}{\sqrt{ca}} \ge 3\sqrt[3]{\frac{abc}{abc}} = 3$

$$\Rightarrow -\left(\frac{c}{\sqrt{ab}}+\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}\right)\leq -3 \quad \textbf{(2)}$$

$$LHS \leq 3+2s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3 \ \textbf{(using (A), (1) (2))}$$

$$\overset{Luenberger}{\leq} 2s\left(\frac{9R}{4\Delta}\right)=\frac{9Rs}{2rs}=\frac{9R}{2r}$$

(proved)

Proof 4 by Adil Abdullayev – Baku – Azerbaidian

$$LHS \leq RHS \Leftrightarrow \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\leq \frac{9R}{2r}\Leftrightarrow$$

$$\Leftrightarrow A:=\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^2\leq \frac{81R^2}{4r^2}$$

$$A \overset{C-B-S}{\leq} 3(a+b+c)3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\leq \frac{81R^2}{4r^2}\Leftrightarrow$$

$$2p\cdot\frac{p^2+r^2+4Rr}{4prR}\leq \frac{9r^2}{4r^2}\Leftrightarrow p^2\leq \frac{9R^3}{2r}-r^2-4Rr$$

$$p^2 \overset{GERRETSEN}{\leq} 4R^2+4Rr+3r^2\leq \frac{9R^3}{2r}-r^2-4Rr \Leftrightarrow 9R^3-8r(R+r)^2\geq 0$$

$$t := \frac{R}{r} \Rightarrow t \geq 2. \quad 9t^3 - 8(t+1)^2 \geq 0 \Leftrightarrow (t-2)(9t^2 + 10t + 4) \geq 0$$

$$\Leftrightarrow EULER$$

Proof 5 by Martin Lukarevski-Stip-Macedonia

Cioplea's inequality. In a triangle ABC let R and r denote the circumradius and inradius respectively. Prove that

$$(\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C})\left(\frac{1}{\sqrt{\sin A}} + \frac{1}{\sqrt{\sin B}} + \frac{1}{\sqrt{\sin C}}\right) \leqslant \frac{9R}{2r}.$$

Solution. Let a, b, c be the sides of the triangle and s its semiperimeter. The inequality is equivalent to

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2 \leqslant \frac{81R^2}{4r^2}.$$

We use the Leuenberger's inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leqslant \frac{\sqrt{3}}{2r}.$$

By Cauchy-Schwarz, we get

$$\begin{aligned}(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2 &\leqslant 3(a+b+c) \cdot 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \\ &\leqslant 18s \cdot \frac{\sqrt{3}}{2r} \\ &\leqslant \frac{81R^2}{4r^2},\end{aligned}$$

which proves the inequality. The last inequality follows from the well-known $s \leqslant 3\sqrt{3}\frac{R}{2}$ and Euler's $R \geqslant 2r$.

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