Let ABC be a triangle with circumradius R and inradius r. Prove that

$$4 \leq \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \leq \left(\frac{R}{r}\right)^2$$

Proposed by George Apostolopoulos –Messalonghi– Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Martin Lukarevski-Skopje-Macedonia

Probar en un triángulo ABC:

$$4 \leq \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \leq \left(\frac{R}{r}\right)^2$$

La desigualdad es equivalente:

$$4 \leq \left(1 + \tan^2\frac{A}{2}\right) + \left(1 + \tan^2\frac{B}{2}\right) + \left(1 + \tan^2\frac{C}{2}\right) \leq \left(\frac{R}{r}\right)^2$$

Desde que:

$$\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \to \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{A}{2}\tan\frac{C}{2} = 1$$

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1 \Leftrightarrow \left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)^2 \geq 3 \Leftrightarrow$$

$$\Leftrightarrow \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3}$$

Además:

$$\tan\frac{A}{2} = \frac{r_a}{p}, \tan\frac{B}{2} = \frac{r_b}{p}, \tan\frac{C}{2} = \frac{r_c}{p}$$

$$\Rightarrow r_a + r_b + r_c = 4R + r$$

$$\left(1 + \tan^2\frac{A}{2}\right) + \left(1 + \tan^2\frac{B}{2}\right) + \left(1 + \tan^2\frac{C}{2}\right) = \left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)^2 + 1 \leq \left(\frac{R}{r}\right)^2$$

$$\Rightarrow \left(\frac{4R+r}{p}\right)^2 + 1 \le \left(\frac{R}{r}\right)^2 \to (4R+r)^2r^2 + p^2r^2 \le p^2R^2 \Rightarrow$$

$$\Rightarrow (4R+r)^2r^2 \le p^2(R^2-r^2)$$

Por la desigualdad de Gerretsen:

$$p^2(R^2-r^2) \ge (16Rr-5r^2)(R^2-r^2) \ge (4R+r)^2r^2$$

$\Rightarrow$ ***Solo es suficiente probar que:*** $(16Rr-5r^2)(R^2-r^2) \ge (4R+r)^2r^2$

$$16R^3r - 5R^2r^2 - 16Rr^3 + 5r^4 \ge (16R^2+8Rr+r^2)r^2$$

$$\Rightarrow 16R^3r - 21R^2r^2 - 24Rr^3 + 4r^4 \ge 0$$

$$\Rightarrow r(16R^3 - 21R^2r - 24Rr^2 + 4r^3) \ge 0$$

$$\Rightarrow r\left(16R^2(R-2r) + 11Rr(R-2r) - 2r^2(R-2r)\right) \ge 0$$

$\Rightarrow r(R-2r)(16R^2+11Rr-2r^2) \ge 0 \Leftrightarrow R \ge 2r$ ***... (Desigualdad e Euler)***

Además: $16R^2+11Rr-2r^2 \ge 64r^2+22r^2-2r^2 = 84r^2 > 0$

Por lo tanto se ha demostrado:

$$4 \le \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \le \left(\frac{R}{r}\right)^2$$ ***... (LQQD)***

Solution 2 by Martin Lukarevski-Skopje-Macedonia

97. In a triangle ABC let R and r denote the circumradius and inradius respectively. Prove that

$$4 \leqslant \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \leqslant \left(\frac{R}{r}\right)^2.$$

Solution. The stronger inequality

$$5 - \frac{2r}{R} \leqslant \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \leqslant 2 + \frac{R}{r}$$

holds. We use the Garfunkel-Bankoff inequality, [1]

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

which by the well-known identity $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$ is equivalent to

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 2 - \frac{2r}{R}.$$

Then

$$\begin{aligned}\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2} &= 3+\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\\ &\geqslant 5-\frac{2r}{R},\end{aligned}$$

which proves the LHS of the inequality. For the RHS we use the inequality

$$(4R+r)^2 \leqslant s^2\left(1+\frac{R}{r}\right),$$

which follows from the Gerretsen's inequality $s^2 \geqslant 16Rr-5r^2$ and Euler's $R \geqslant 2r$, combined with the identity

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{(4R+r)^2-2s^2}{s^2}.$$

We have

$$\begin{aligned}\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2} &= 1+\left(\frac{4R+r}{s}\right)^2\\ &\leqslant 2+\frac{R}{r},\end{aligned}$$

and we are done.

References

[1] Problem 825 (proposed by J. Garfunkel, solution by L. Bankoff) *Crux Math.* 9 (1983), 79 and 10 (1984), 168

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