SEQUENTIALLY CONVERGENT MAPPINGS AND COMMON FIXED POINTS OF MAPPINGS IN 2-BANACH SPACES

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1. INTRODUCTION

In 1968 White ([3]) introduces 2-Banach spaces. 2-Banach spaces are being studied by several authors, and certain results can be seen in [8]. Further, analogously as in normed space P. K. Hatikrishnan and K. T. Ravindran in [6] are introducing the term contractive mapping to 2-normed space as follows.

Definition 1 ([6]). Let $(L,\|\cdot,\cdot\|)$ be a real vector 2-normed space. The mapping $S:L\to L$ is contraction if there is $\lambda\in[0,1)$ such that $\|Sx-Sy,z\|\leq\lambda\|x-y,z\|$, for all $x,y,z\in L$.

Regarding contractive mapping Hatikrishnan and Ravindran in [6] proved that contractive mapping has a unique fixed point in closed and restricted subset of 2-Banach space. Further, in [1], [4], [5] and [7] are proven more results related to fixed points on contractive mapping of 2-Banach spaces, and in [7] are proven several results for common fixed points of contractive mapping defined on the same 2-Banach space. In our further considerations, we will give some generalizations of the above results for common fixed points of mapping defined on the same 2-Banach space. Thus, the mentioned generalizations we will do it with the help of so-called sequentially convergent mappings which are defined as follows.

Definition 2. Let $(L,||\cdot,\cdot||)$ be a 2-normed space. A mapping $T: L \to L$ is said to be sequentially convergent if, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent.

2. Common fixed points on mapping by the Kannan type

Theorem 1. Let $(L, \|\cdot,\cdot\|)$ be a 2- Banach space, $S_1, S_2 : L \to L$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0$, $\gamma \geq 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1 x - TS_2 y, z \| \leq \alpha (\| Tx - TS_1 x, z \| + \| Ty - TS_2 y, z \|) + \gamma \| Tx - Ty, z \|, \qquad (1)$$

for each $x, y, z \in L$, then S_1 and S_2 have a unique common fixed point $z \in L$. ■

Corollary 1. Let $(L,\|\cdot,\cdot\|)$ be a 2- Banach space, $S_1, S_2 : L \to L$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0$, $\gamma \geq 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1x - TS_2y, z \| \leq \alpha \frac{\|Tx - TS_1x, z\|^2 + \|Ty - TS_2y, z\|^2}{\|Tx - TS_1x, z\| + \|Ty - TS_2y, z\|} + \gamma \| Tx - Ty, z \|,$$

for each $x, y, z \in L$, $z \neq 0$, then S_1 and S_2 have a unique common fixed point $z \in L$.

Proof. From inequality of condition following inequality (1). Now the assertion follows from Theorem 1. ■

Corollary 2. Let $(L,\|\cdot,\cdot\|)$ be a 2- Banach space, $S_1,S_2:L\to L$ and mapping $T:L\to L$ is continuous, injection and sequentially convergent. If $0<\lambda<1$ and

$$\|TS_1x-TS_2y,z\|\leq\lambda\cdot\sqrt[3]{\|Tx-TS_1x,z\|\cdot\|Ty-TS_2y,z\|\cdot\|Tx-Ty,z\|},$$

for each $x,y,z\in L$, then S_1 and S_2 have a unique common fixed point $z\in L$.

Proof. From the inequality between the arithmetic and geometric mean follows that

$$d(TS_1x,TS_2y)\leq\frac{\lambda}{3}(d(Tx,TS_1x)+d(Ty,TS_2y)+\beta d(Tx,Ty)).$$

Now the assertion follows from Theorem 1 for $\alpha=\gamma=\frac{\lambda}{3}$. ■

Corollary 3. Let $(L, \|\cdot,\cdot\|)$ be a 2- Banach space, $S_1^p, S_2^q : L \to L$, $p, q \in \mathbf{N}$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0, \gamma \geq 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1^p x - TS_2^q y, z \| \leq \alpha(\| Tx - TS_1^p x, z \| + \| Ty - TS_2^q y, z \|) + \gamma \| Tx - Ty, z \|,$$

for each $x, y, z \in L$. Then S_1 and S_2 have a unique common fixed point $u \in L$. ■

Remark 1. Mapping $T : L \to L$ determined by $Tx = x, x \in L$ is sequentially convergent. Therefore, if in theorem 1 and the corollaries 1, 2 and 3 we take that $Tx = x$ appropriate following the accuracy of Theorem 4 and corollaries 6, 7 and 8, [7].

3. COMMON FIXED POINTS OF MAPPINGS OF CHATTERJEA TYPE

Theorem 2. Let $(L, \|\cdot,\cdot\|)$ be a 2- Banach space, $S_1, S_2 : L \to L$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0$, $\gamma \geq 0$, are such that $2\alpha + \gamma < 1$ and

$$\| TS_1 x - TS_2 y, z \| \leq \alpha (\| Tx - TS_2 y, z \| + \| Ty - TS_1 x, z \|) + \gamma \| Tx - Ty, z \|, \qquad (4)$$

for each $x, y, z \in L$, then S_1 and S_2 have a unique common fixed point $u \in L$. ■

Corollary 4. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1,S_2:L\to L$ and the mapping $T:L\to L$ is continuous, injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$ are such that $2\alpha+\gamma<1$ and

$$\|TS_1x-TS_2y,z\|\leq\alpha\frac{\|Tx-TS_2y,z\|^2+\|Ty-TS_1x,z\|^2}{\|Tx-TS_2y,z\|+\|Ty-TS_1x,z\|}+\gamma\|Tx-Ty,z\|,$$

for each $x,y,z\in L$, $z\neq 0$, then S_1 and S_2 have a unique common fixed point $u\in L$.

Proof. From inequality of condition follows inequality (4). Now the assertion follows from Theorem 2.■

Corollary 5. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1, S_2 : L \to L$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $0 < \lambda < 1$ and

$$\| TS_1x - TS_2y, z \| \leq \lambda \cdot \sqrt[3]{\| Tx - TS_2y, z \| \cdot \| Ty - TS_1x, z \| \cdot \| Tx - Ty, z \|} ,$$

for each $x, y, z \in L$, then S_1 and S_2 have a unique common fixed point $z \in L$.

Proof. From the inequality between the arithmetic and geometric mean follows that

$$d(TS_1x, TS_2y) \leq \frac{\lambda}{3}(d(Tx, TS_2y) + d(Ty, TS_1x) + d(Tx, Ty)) .$$

Now the assertion follows from Theorem 2 for $\alpha = \gamma = \frac{\lambda}{3}$. ■

Corollary 6. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1^p, S_2^q : L \to L$, $p,q \in \mathbf{N}$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0, \gamma \geq 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1^p x - TS_2^q y, z \| \leq \alpha(\| Tx - TS_2^q y, z \| + \| Ty - TS_1^p x, z \|) + \gamma \| Tx - Ty, z \|,$$

for each $x, y, z \in L$. Then S_1 and S_2 have a unique common fixed point $u \in L$.

Proof. The proof is identical to the proof of the corollary 5. ■

Remark 2. The mapping $T : L \to L$ determined by $Tx = x, x \in L$ is sequentially convergent. Therefore, if in Theorem 2 and corollaries 4, 5 and 6 we take $Tx = x$, follows the correctness of Theorem 5 and corollaries 9, 10 и 11, [7].

4. COMMON FIXED POINTS OF MAPPINGS OF KOPARDE-WAGHMODE TYPE

Theorem 3. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1,S_2:L\to L$ and mapping $T:L\to L$ is continuous, injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$, $2\alpha+\gamma<1$ and

$$\|TS_1x-TS_2y,z\|^2\leq\alpha(\|Tx-TS_1x,z\|^2+\|Ty-TS_2y,z\|^2)+\gamma\|Tx-Ty,z\|^2, \quad (6)$$

for each $x,y,z\in L$, then S_1 and S_2 have a unique common fixed point $u\in L$. ■

Corollary 7. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1^p, S_2^q : L \to L$, $p,q \in \mathbf{N}$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0, \gamma \geq 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1^p x - TS_2^q y, z \|^2 \leq \alpha(\| Tx - TS_1^p x, z \|^2 + \| Ty - TS_2^q y, z \|^2) + \gamma \| Tx - Ty, z \|^2,$$

for each $x, y, z \in L$. Then S_1 and S_2 have a unique common fixed point $u \in L$.

Proof. The proof is identical to the proof of the corollary 6. ■

Remark 3. The mapping $T : L \to L$ determined by $Tx = x, x \in L$ is sequentially convergent. Therefore, if in Theorem 3 and corollary 7 we take $Tx = x$, it follows the correctness of Theorem 6 and corollary 12, [7].

References

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