the whole of his career at Bridgewater State University, Massachusetts. He composed over 100 problems for a range of mathematical journals.

Correct solutions were received from: R. G. Bardelang, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, F. Hunt, L. Ma Li & A. Plaza, J. A. Mundie, G. Strickland, K. B. Subramaniam, I. Timmins, R. F. Tindall and the proposer T. Moore.

99.B (Michael Fox)

Find the smallest triangle whose sides are consecutive integers and whose area is an integer greater than 10^9.

Answer: The smallest triangle has sides 140 451, 140 452, 140 453 with area 8 541 939 510.

Almost all respondents used standard methods to analyse the Pell equation underlying this problem.

If the sides of the triangle are $N - 1, N, N + 1$, then its area A is given by $A = \frac{1}{4}\sqrt{3N^2(N^2 - 4)}$ so that $16A^2 = 3N^2(N^2 - 4)$. It follows that $N = 2x$ is even and that $x^2 - 1 = 3y^2$ with $A = 3xy$. The solutions of $x^2 - 3y^2 = 1$ are generated from the fundamental solution $x_1 = 2, y_1 = 1$ by the recurrence relations

$$x_{n+1} = 2x_n + 3y_n, \qquad y_{n+1} = x_n + 2y_n \tag{A}$$

$$\text{or} \qquad x_{n+1} = 4x_n - x_{n-1}, \qquad y_{n+1} = 4y_n - y_{n-1} \tag{B}$$

which solve to give

$$x_n = \frac{1}{2}\left[\left(2 + \sqrt{3}\right)^n + \left(2 - \sqrt{3}\right)^n\right], y_n = \frac{1}{2\sqrt{3}}\left[\left(2 + \sqrt{3}\right)^n - \left(2 - \sqrt{3}\right)^n\right] \tag{C}$$

which may also be written as

$$x_n = \cosh n\theta, \ y_n = \frac{1}{\sqrt{3}} \sinh n\theta \tag{D}$$

where $\theta = \ln\left(2 + \sqrt{3}\right)$. Any of (A) — (D) may be used to hunt down the required solution $n = 9$ with $x_9 = 70226$. $y_9 = 40545$ giving the triangle in the answer above. The proposer, Michael Fox, noted that the difference in area between this triangle and an equilateral triangle of side 140452 is almost exactly $\frac{\sqrt{3}}{2}$.

Stan Dolan commented that, in the literature, triangles with side-lengths $N - 1$, N, $N + 1$ are sometimes known as 'almost equilateral': a useful reference is the *Wikipedia* entry on Heronian triangles.

Correct solutions were received from: R. G. Bardelang, M. Bataille, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, P. F. Johnson, M. Lukarevski, J. Moore, J. A. Mundie, A. C. Robin, G. Strickland, K. B. Subramaniam, E. Swylan, I. Timmins, R. F. Tindall and the proposer M. Fox.