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ON A NUMERICAL SOLUTION OF THE LAPLACE EQUATION

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Abstract. The Laplace equation in three variables can be reduced to three ODEs by means of the Fourier method. For the cases when the exact solution of the obtained ODEs does not exist, or it is complicated, we apply the wavelet-Galerkin method. We use suitable wavelets or scaling functions that allow finding the numerical solutions of the three equations, which will form the solution of the Laplace equation.

1. Introduction

Wavelets are well localized, oscillatory functions which provide a basis of $L^2(\mathbb{R})$ and can be modified to a basis of $L^2([a,b])$, where $[a,b]$ is a bounded domain. Their well localization allows local variations of the problem to be analyzed at various levels of resolution. The concepts of wavelet theory were provided by Meyer, Mallat, Daubechies and many others, [4, 5, 10]. Wavelets have several properties that are especially useful for representing solutions of differential equations, such as orthogonality, compact support and exact representation of polynomials of a certain degree. Multiresolution analysis using orthogonal and compactly supported wavelets has been successfully applied in numerical simulation. So, the use of wavelet-based numerical schemes has become increasingly popular in the last two decades. The wavelet-Galerkin method has emerged as an accurate and efficient means of approximating the solution of the ordinary and the partial differential equations. It is an improvement over the standard Galerkin methods by using a compactly supported orthogonal functional basis, [1, 2, 6, 9, 11, 12, 14].

The aim of this paper is to find a numerical solution for the Laplace PDE. After reducing it to three ODEs by means of the Fourier method, we apply the wavelet-Galerkin method to the obtained ODEs. We use suitable wavelets or scaling functions that allow

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finding the numerical solutions of the three equations, which will form the solution of the Laplace equation.

The outline of this paper is as follows: In Section 2 we summarize some basics of the wavelet analysis. In Section 3 we describe the wavelet-Galerkin method for the ODE at arbitrary level. The transformation of the Laplace PDE in three variables to the three ODEs by means of the Fourier method is illustrated in Section 4. And, in the last section, the wavelet-Galerkin method is applied to the three ODEs obtained in Section 4.

2. Wavelet preliminaries

An oscilatory function $\psi \in L^2(R)$ with zero mean, i.e.

$$\int_{\Re} \psi(t)dt = 0;$$

is called a *wavelet* (*mother wavelet*) if it has the following desirable properties:
1. Smoothness: ψ is $n-$times differentiable and their derivatives are continuous;
2. It satisfies the admissibility condition:

$$\int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty;$$

3. Localization: ψ is well localized both in time and frequency domains, i.e. ψ and its derivatives must decay very rapidly.

The goal of the multiresolution analysis (MRA) is to develop representations of a function $f(x)$ at various levels of resolution. This can be achieved by expanding the given function in terms of basis functions which can be scaled to give multiple resolutions of the original function. In order to develope a multilevel representation of a function in $L^2(\mathbb{R})$, we require sequence of embedded closed subspases $\{V_j\}$ such that

$$\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2(\mathbb{R})$$

with the following properties:
1. $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;
2. $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
3. The embedded subspaces are related by the scaling law,

$$f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1};$$

4. Each subspace is spanned by integer translates of a single function $f(x)$ such that

$$f(t) \in V_j \Leftrightarrow f(t-k) \in V_j, \forall k \in \mathbb{Z};$$

5. There exists a function ϕ (called *scaling function* or *father wavelet*) such that $\phi_{j,k}(t) = 2^{-j/2}\phi(2^{-j}t - k)$, $k \in \mathbb{Z}$, constitute orthonormal basis for corresponding subspace V_j.

The scaling functions often have significant properties which make them more useful then some other functions:
1. Compact support;

2. The area under the scaling function is unity,

$$\int_{-\infty}^{\infty} \phi(t)dt = 1;$$

3. Their translations are orthogonal,

$$\int_{-\infty}^{\infty} \phi(t-n)\phi(t-k)dt = \delta_{k,n},$$

where $\delta_{n,k}$ is the Kronecker delta defined by

$$\delta_{n,k} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases};$$

4. They satisfy a dilation equation

$$\phi(t) = \sum_{k \in \mathbb{Z}} a_k \phi(2t-k), \tag{2.1}$$

where a_k are real coefficients and $a_k \neq 0$ for only finitely many $k \in \mathbb{Z}$ (the number of nonzero coefficients a_k in the series (2.1) is denoted by L).

Some of these properties, such like compact support and unit integral, are required for using the scaling functions family in numerical methods, especially for boundary value problems. Others, like orthogonality, are desirable but not so necessary.

If $\phi \in L^2(\mathbb{R})$ is compactly supported scaling function of MRA, one can construct the wavelet ψ such that $\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t-k)$, $j,k \in \mathbb{Z}$, constitute an orthonormal basis for $L^2(\mathbb{R})$. A complete wavelet theory can be found in [4, 5, 7, 8, 10].

3. Wavelet-Galerkin method for ordinary differential equations

Since we apply the wavelet-Galerkin method to homogeneous differential equations of second order, in this section, we explain the application of the method to this type of equation,

$$g(t)u''(t) + h(t)u'(t) + r(t)u(t) = 0, \quad t \in [a,b], \tag{3.1}$$

with the boundary conditions

$$u(a) = c, u(b) = d, \tag{3.2}$$

where $g(t), h(t)$ and $r(t)$ are real-valued continuous functions on $[a,b]$. This method also can be applied to other type of ODEs on a similar way.

Let the solution $u(t)$ of the equation (3.1) be approximated by its j-th level scaling function expansion on the interval (a,b),

$$u_j(t) = \sum_{k=1-L}^{2^j} c_k \phi_{j,k}(t), k \in \mathbb{Z}, \tag{3.3}$$

where ϕ is a scaling function of MRA and c_k are unknown coefficients that should be determined. It is clear that the larger integer j is used, the more accurate solution is obtained.

The boundaries of the support of $u_j(t)$ given by (3.3) are $\dfrac{1-L}{2^j}$ and $\dfrac{L-1+2^j}{2^j}$. Subsequently, the original boundaries a and b are now changed to fictitious boundaries, i.e. the boundaries a and b are extended by an amount $a - \dfrac{1-L}{2^j}$ and $\dfrac{L-1+2^j}{2^j} - b$, respectively without affecting the solution within $[a, b]$, so the affected solution is within the intervals $\left[\dfrac{1-L}{2^j}, a\right]$ and $\left[b, \dfrac{L-1+2^j}{2^j}\right]$.

Substituting (3.3) in the differential equation (3.1), we get

$$\begin{aligned} g(t)\frac{d^2}{dt^2}\sum_{k=1-L}^{2^j} c_k 2^{-j/2}\phi(2^{-j}t-k) + h(t)\frac{d}{dt}\sum_{k=1-L}^{2^j} c_k 2^{-j/2}\phi(2^{-j}t-k) + \\ + \quad r(t)\sum_{k=1-L}^{2^j} c_k 2^{-j/2}\phi(2^{-j}t-k) = 0. \end{aligned}$$

To determinate the coefficients c_k, we take inner product with $\phi_{j,n}(t) = 2^{-j/2}\phi(2^{-j}t-n)$, $n \in \{1-L, ..., 2^j\}$ and obtain

$$\begin{aligned} &\sum_{k=1-L}^{2^j} c_k \int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}} g(t)2^{-j}\phi''(2^{-j}t-k)\phi(2^{-j}t-n)dt \\ + &\sum_{k=1-L}^{2^j} c_k \int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}} h(t)2^{-j}\phi'(2^{-j}t-k)\phi(2^{-j}t-n)dt \\ + &\sum_{k=1-L}^{2^j} c_k \int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}} r(t)2^{-j}\phi(2^{-j}t-k)\phi(2^{-j}t-n)dt = 0, \end{aligned}$$

i.e.

$$\sum_{k=1-L}^{2^j} c_k t_{n,k} = 0, \tag{3.4}$$

where

$$\begin{aligned} t_{n,k} &= \Omega_{n-k} + a_{n,k} + s_{n,k}, \\ \Omega_{n-k} &= \int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}} 2^{-j}g(t)\phi''(2^{-j}t-k)\phi(2^{-j}t-n)dt, \\ a_{n,k} &= \int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}} 2^{-j}h(t)\phi'(2^{-j}t-k)\phi(2^{-j}t-n)dt, \\ s_{n,k} &= \int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}} 2^{-j}r(t)\phi(2^{-j}t-k)\phi(2^{-j}t-n)dt. \end{aligned}$$

Using the boundary conditions (3.2), we obtain these two equations

$$u_j(a) = \sum_{k=1-L}^{2^j} c_k\phi_{j,k}(a) = c, \tag{3.5}$$

and

$$u_j(b) = \sum_{k=1-L}^{2^j} c_k\phi_{j,k}(b) = d. \tag{3.6}$$

The equations (3.5) and (3.6) give the relation between the coefficients c_k, $k \in \{1-L, ..., 2^j\}$. Now, we eliminate the first and the last equation of the system (3.4) and we put the equations (3.5) and (3.6) on their places. So, we get the following matrix equation

$$TC = B,$$

where

$$T = \begin{bmatrix} \phi_{j,1-L}(a) & \phi_{j,2-L}(a) & \phi_{j,3-L}(a) & \dots & \phi_{j,2^j}(a) \\ t_{2-L,1-L} & t_{2-L,2-L} & t_{2-L,3-L} & \dots & t_{2-L,2^j} \\ t_{3-L,1-L} & t_{3-L,2-L} & t_{3-L,3-L} & \dots & t_{3-L,2^j} \\ \vdots & & & & \\ t_{2^j-1,1-L} & t_{2^j-1,2-L} & t_{2^j-1,3-L} & \dots & t_{2^j-1,2^j} \\ & & & & \\ \phi_{j,1-L}(b) & \phi_{j,2-L}(b) & \phi_{j,3-L}(b) & \dots & \phi_{j,2^j}(b) \end{bmatrix}$$

and

$$C = [c_{1-L} \; c_{2-L} \; \dots \; c_{2^j}]^T, B = [c \; 0 \; \dots \; 0 \; d]^T.$$

By Gaussian elimination algorithm we get the coefficients c_k, $k \in \{1-L, \cdots, 2^j\}$ and the approximate solution u_j given by (3.3).

4. Trasformation of the Laplace equation by Fourier method

The famous Laplace equation

$$\triangle u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

transformed in polar coordinates using the substitutions

$$x = r \sin\theta \cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

has the form

$$\triangle u \equiv \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0. \tag{4.1}$$

A Fourier method, which subsumes that the unknown function $u = u(r, \theta, \varphi)$ can be represented as a product of three functions of one variable, $u(r, \varphi, \theta) = R(r)\Phi(\varphi)\Theta(\theta)$, can be used for reducing the last equation on three ODEs. Assuming that the functions $R(r), \Phi(\varphi)$ and $\Theta(\theta)$ are twice differentiable functions, the equation (4.1) takes the form

$$\Phi\Theta\frac{d}{dr}(r^2R') + R\Phi\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Theta') + \frac{1}{\sin^2\theta}R\Theta\Phi = 0,$$

or, after division by $R\Phi\Theta$, the form

$$\frac{1}{R}\cdot\frac{d}{dr}(r^2R') + \frac{1}{\Theta}\cdot\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}(\sin\theta\cdot\Theta') + \frac{1}{\sin^2\theta}\cdot\frac{\Phi''}{\Phi} = 0.$$

If we rearrange it,

$$\frac{1}{R}\cdot\frac{d}{dr}(r^2R') = -\left[\frac{1}{\Theta}\cdot\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}(\sin\theta\cdot\Theta') + \frac{1}{\sin^2\theta}\cdot\frac{\Phi''}{\Phi}\right],$$

we can conclude that both sides of the equation are constant, which might be denoted by λ:

$$\frac{1}{R}\cdot\frac{d}{dr}(r^2R') = \lambda, \qquad \frac{1}{\Theta}\cdot\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}(\sin\theta\cdot\Theta') + \frac{1}{\sin^2\theta}\cdot\frac{\Phi''}{\Phi} = -\lambda. \tag{4.2}$$

After rewriting the second equation of (4.2) in the form

$$\frac{1}{\Theta}\cdot\sin\theta\cdot\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\Theta}{d\theta}\right) + \lambda\sin^2\theta = -\frac{\Phi''}{\Phi}, \tag{4.3}$$

it is obvious that both sides in (4.3) are also constant, let's denote this constant by μ,

$$-\frac{\Phi''}{\Phi} = \mu, \qquad \frac{1}{\Theta}\cdot\sin\theta\cdot\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\Theta}{d\theta}\right) + \lambda\sin^2\theta = \mu. \tag{4.4}$$

The first equation from (4.2) and the both equations from (4.4) form a system of three ODEs, whose solution will form the solution of (4.1), according to Fourier's method.

The first ODE is the Cauchy-Euler equation obtained from (4.2),

$$r^2R'' + 2rR' - \lambda R = 0, \tag{4.5}$$

whose exact solution is of the form $R = r^\alpha$ and according to the literature (see [13]) it can be obtained for $\lambda = n(n+1)$,

$$R(r) = C_1r^n + C_2\frac{1}{r^{n+1}}.$$

Note that, often only the function $R(r) = C_1r^n$ is taken as a solution, which is the case when the interval of consideration contains $r = 0$ (otherwise, the solution will contradict the condition for twice differentiability of $R(r)$).

The second ODE, obtained from (4.4) is the following homogeneous Sturm-Liouville equation

$$\Phi'' + \mu\Phi = 0, \tag{4.6}$$

whose nonzero general solution can be obtained in its explicit form,

$$\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi,$$

for $\mu = m^2$, where $m = 1, 2, \dots$ ([3]).

The third ODE, obtained from (4.4) is also a homogeneous Sturm-Liouville equation,

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \Theta\left(\lambda\sin^2\theta - \mu\right) = 0, \tag{4.7}$$

Taking into consideration that the constants keep the assigned values, $\lambda = n(n+1)$ and $\mu = m^2$, the explicit form of the solution of (4.7) is given with

$$P_{n,m} = (1-x^2)^{\frac{m}{2}}\frac{d^mP_n(x)}{dx^m} = \frac{(1-x^2)^{\frac{m}{2}}}{n!2^n}\frac{d^{n+m}}{dx^{n+m}}[(x^2-1)^n],$$

where

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}[(x^2-1)^n]$$

are Legendre polynomials, and $x = \cos\theta$ (see [13]).

5. Numerical results

In this section we give the wavelet-Galerkin solutions of the ODEs in which the Laplace equation is reduced, for different values of the parameters (for some values the exact solution exists and for some, it doesn't exist). For a scaling function we use the quadratic B-spline (5.1) and the sinc function (5.2).

$$\phi(t)=\left\{\begin{array}{rl} \frac{1}{2}t^2, & t\in[0,1]\\ -t^2+3t-\frac{3}{2}, & t\in[1,2]\\ \frac{1}{2}t^2-3t+\frac{9}{2}, & t\in[2,3]\\ 0, & t\notin[0,3]\end{array}\right. , \tag{5.1}$$

$$\phi_s(t)=\left\{\begin{array}{rl} \frac{\sin(\pi t)}{\pi t}, & t\neq 0\\ 1, & t=0\end{array}\right. . \tag{5.2}$$

It is known that the sinc function satisfies an infinite equation of dilatation (2.1), while the quadratic B-spline function satisfies the following dilatation equation

$$\phi(t)=\frac{1}{4}\phi(2t)+\frac{3}{4}\phi(2t-1)+\frac{3}{4}\phi(2t-2)+\frac{1}{4}\phi(2t-3).$$

Since we want to compare the obtained numerical B-spline and sinc solutions, we take $L=4$ for both of them. It is clear that the more accurate solutions can be obtained using sinc function, by taking larger integer L.

We consider the equation (4.5) with the boundary conditions $R(1)=1, R(3)=0$. For $\lambda=0$ it has the form

$$r^2R''+2rR'=0.$$

Its exact solution and approximate solution using the quadratic B-spline at zero level are

$$R(r)=-\frac{1}{2}+\frac{3}{2r}$$

and

$$R_0(r)=\left\{\begin{array}{rl} \frac{690}{431}\phi(r+1)+\frac{172}{431}\phi(r), & r\in[1,2]\\ \frac{172}{431}\phi(r), & r\in[2,3]\end{array}\right. , \tag{5.3}$$

respectively, and they are shown on Figure 1.

The numerical solutions of (4.5), for $\lambda=1,3,4$, obtained using the B-spline function at zero level, are shown on Figure 2.

Now, we consider the equation (4.6), for $\mu=4$ and the boundary conditions $\Phi(0)=1$, $\Phi(\frac{\pi}{4})=-1$. In this case, its exact solution is

$$\Phi(\varphi)=\cos(2\varphi)-\sin(2\varphi).$$

Using B-spline function at the zeroth and first level, we obtain the following approximate solutions

$$\Phi_0(\varphi)=2.61431\phi(\varphi+2)-0.614306\phi(\varphi+1)-2.10591\phi(\varphi),\ \varphi\in[0,\frac{\pi}{4}], \tag{5.4}$$

and

$$\Phi_1(\varphi)=\frac{1}{\sqrt{2}}\left(5.708740\phi\left(\frac{\varphi}{2}+2\right)-2.8803\phi\left(\frac{\varphi}{2}+1\right)-4.407829\phi\left(\frac{\varphi}{2}\right)\right),$$

$\varphi\in\left[0,\frac{\pi}{4}\right]$, respectively.

If we use the sinc function at zero level, we obtain the solution

$$\Phi_{0,s}(\varphi) = c_3\phi_1(\varphi+3) + c_2\phi_1(\varphi+2) + c_1\phi_1(\varphi+1) + c_0\phi_1(\varphi) + c_{-1}\phi_1(\varphi-1), \quad \varphi \in [0, \frac{\pi}{4}],$$

where $c_3 = -1.061294$, $c_2 = 0.88210$, $c_1 = 0.45482$, $c_0 = -1.08518$, $c_{-1} = 1.25899$.

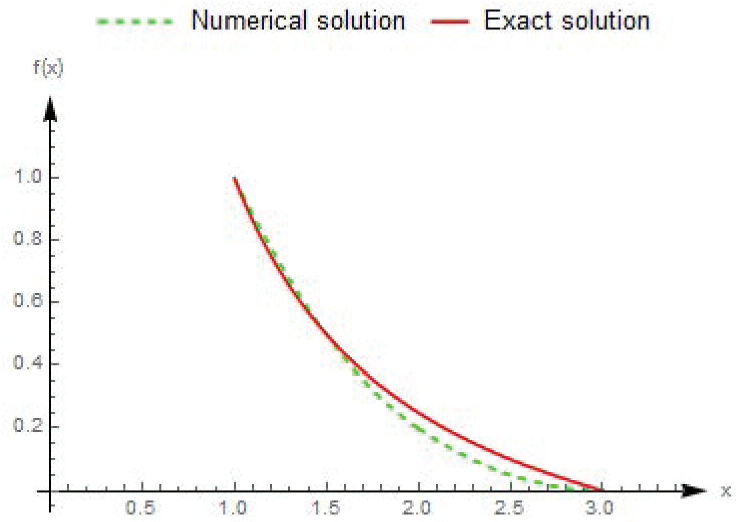

FIGURE 1. Exact and numerical B-spline solution of the equation (4.5), for $\lambda = 0$

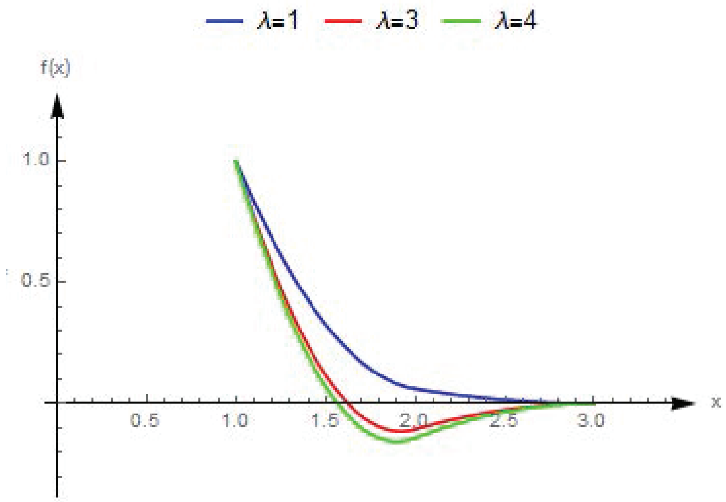

FIGURE 2. Numerical B-spline solutions of the equation (4.5), for $\lambda = 1, 3, 4$

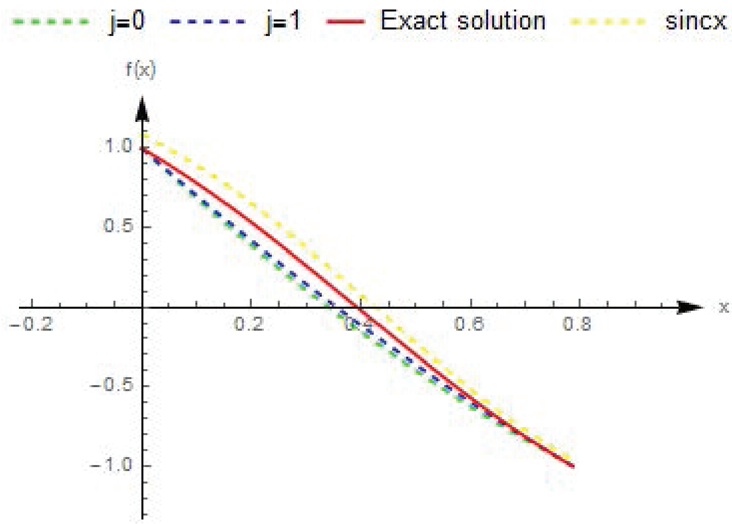

FIGURE 3. Exact and numerical B-spline and sinc solutions of the equation (4.6), for $\mu = 4$

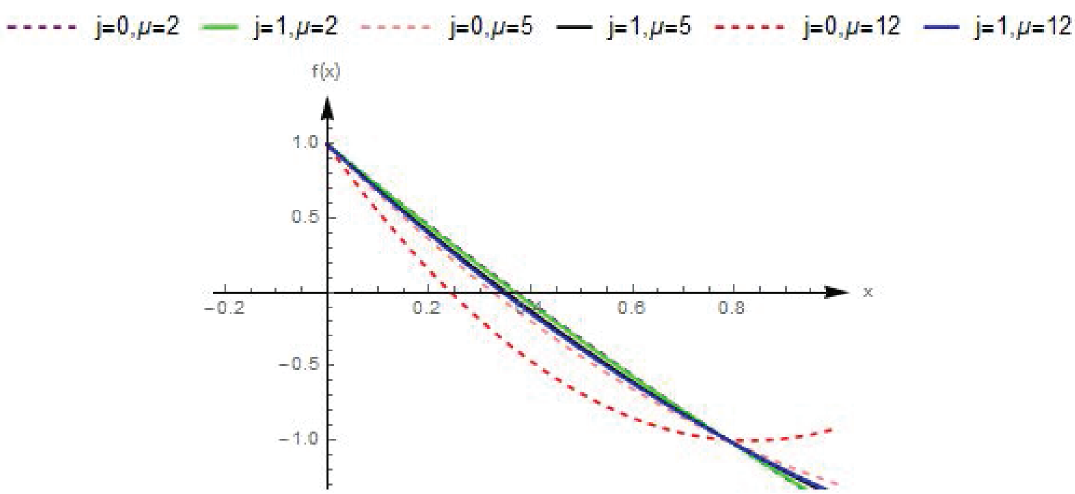

FIGURE 4. Comparison of numerical solutions at zeroth and first level of the equation (4.6), for several values of parameter μ

The solutions of the equation (4.6) are shown on Figures 3 and 4. As expected, the approximate B-spline solution at first level is better then the approximate solution at zeroth level obtained by B-spline or sinc function (see Fig.3). Fig. 4 presents the numerical B-spline solutions obtained for $\mu = 2, 5, 12$, values for which the exact solution does not exist. We can conclude that the numerical solutions for a different values of parameter μ at first level are closer one to the other then the numerical solutions at zero level.

Approximate solutions for the equation (4.7) are obtained on a similar way. For the pairs of parameters: $\lambda = 0, \mu = 2$ and $\lambda = 1, \mu = 4$, the exact solution does not exists. We found the numerical B-spline solutions at zero level, presented on Figure 6.

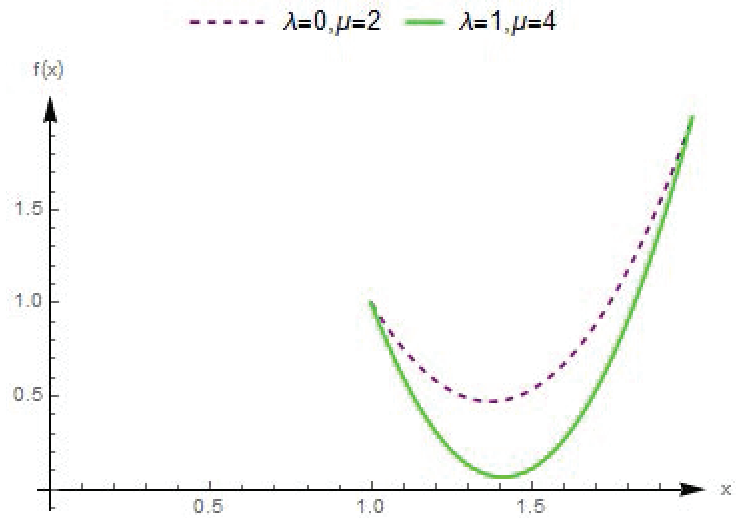

FIGURE 5. Numerical B-spline solutions for the equation (4.7), for $\lambda = 0, \mu = 2$ and $\lambda = 1, \mu = 4$

Now, the numerical solution of the Laplace equation at zero level using B-spline function may be obtained by the formula

$$U_0(r,\phi,\theta) = R_0(r)\Phi_0(\phi)\Theta_0(\theta), \quad r \in [1,3], \phi \in [0,\pi/4], \theta \in [1,2], \tag{5.5}$$

where $R_0(r)$ and $\Phi_0(\phi)$ are given by (5.3) and (5.4), respectively, while $\Theta_0(\theta)$ is the approximate solution of the equation (4.7), with $\lambda = 0, \mu = 4$ and boundary conditions $\Theta(1) = 1, \Theta(2) = 2$. It can be obained by the formula

$$\Theta_0(\theta) = 3.64596\phi(\theta+1) - 1.64596\phi(\theta) + 5.64596\phi(\theta-1), \quad \theta \in [1,2].$$

From the range of the variables r, ϕ and θ, we choose three values $r = 1.3$, $\theta = 1.5$ and $\phi = 0.6$ and get the projections of the numerical solution (5.5) of the LPDE onto $r = 1.3$, $\theta = 1.5$ and $\phi = 0.6$, respectively (see Figure 6, 7 and 8).

6. CONCLUSION

In this paper we find the approximate solution of the Laplace PDE of third order, applying the wavelet-Galerkin method to the three ODEs in which the Laplace equation is reduced. The ODEs depend on two parameters, λ and μ. Varying their values, we consider several cases. In the cases where the ODEs have exact solution we give it, find the numerical solution and compare them (see Fig.1 and Fig.3). In the cases where the exact solution does not exist we find the numerical solutions for different parameters at different levels (see Fig.2, Fig.4, Fig.6). From the obtained results we can conclude that at higher level the wavelet-Galerkin method gives better approximate solutions. At the end, we resume the numerical solution of the Laplace equation.

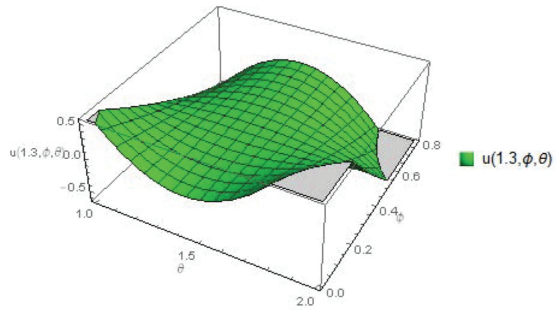

FIGURE 6. Projection of the numerical B-spline solution of the LPDE onto $r = 1.3$

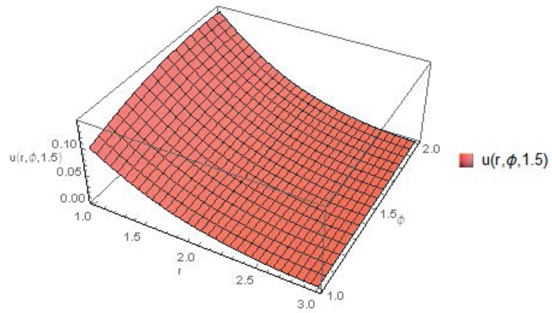

FIGURE 7. Projection of the numerical B-spline solution of the LPDE onto $\theta = 1.5$

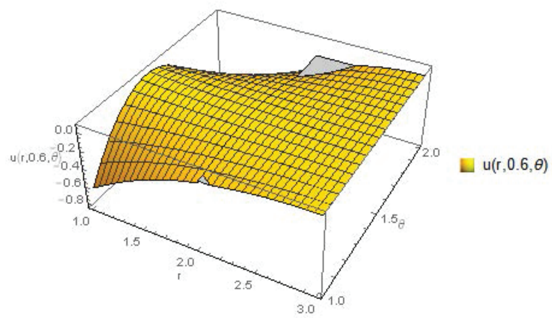

FIGURE 8. Projection of the numerical B-spline solution of the LPDE) onto $\phi = 0.6$

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