Serdica
Mathematical Journal
Сердика
Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **41** (2015), 35–48

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

SHIFT OPERATORS

M. S. Anoussis

Communicated by I. G. Todorov

Abstract. Shift operators play an important role in different areas of Mathematics such as Operator Theory, Dynamical Systems and Compex Analysis. In these lectures we discuss basic properties of these operators. We present Beurling's Theorem which describes the invariant subspaces of the shift. The structure of the C^*-algebra generated by the shift is described. We also indicate how the shift operators appear in the analysis of isometries on a Hilbert space: Wold decomposition and Coburn's theorem.

1. Introduction. Shift operators play an important role in different areas of Mathematics. They occur naturally in Operator Theory, in Dynamical Systems and in Compex Analysis. In these lectures we discuss the following topics: In section 3 we present Beurling's Theorem which describes the invariant subspaces of the shift. In section 4 we analyse the C^*-algebra generated by the shift. In the last section we prove Wold decomposition which essentially shows

2010 *Mathematics Subject Classification:* Primary 46L05; Secondary 47A15, 47B35.
Key words: shift operator, Toeplitz algebra, Wold decomposition, invariant subspace.

that any isometry on a Hilbert space is built from a unitary operator and a shift of some multiplicity. Finally we present Coburn's Theorem which shows that the C^*-algebra generated by a proper isometry is isomorphic to the C^*-algebra generated by the shift.

If H is a Hilbert space we will denote by $\mathcal{B}(H)$ the algebra of bounded linear operators on H. We will denote by $\mathbb{T}$ the circle group, that is the multiplicative group of complex numbers of modulus 1.

2. Spectral properties of the shift. Let H be a separable Hilbert space. Consider an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of H.

The *shift operator* S is the operator on H defined by $Se_n = e_{n+1}$ for $n = 0, 1, 2, \ldots$. The adjoint operator S^* satisfies $S^*e_n = e_{n-1}$ for $n = 1, 2, 3, \ldots$ and $S^*e_0 = 0$.

With respect to the basis $\{e_n\}_{n=0}^{\infty}$ the matrix of S is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the matrix of S^* is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that the operator S is $1-1$ but not onto, and the operator S^* is onto but not $1-1$. The operator SS^* is the orthogonal projection on the subspace spanned by the vectors $e_n : n = 1, 2, \ldots$ and $S^*S = I$.

An operator T in $\mathcal{B}(H)$ is an *isometry* if $\|Tx\| = \|x\|$ for all $x \in H$.

Theorem 2.1.

(1) *The shift operator is a non-unitary isometry.*

(2)
$$\bigcap_{n=0}^{\infty} S^n(H) = \{0\}.$$

(3) *For every* $x \in H$ *we have* $(S^*)^n x \to 0$.

Theorem 2.2. *The spectrum of* S *is equal to the set* $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

P r o o f. Let $\lambda \in \mathbb{C}$, $|\lambda| < 1$. Then

$$S^* \left(\sum_{n=0}^{\infty} \lambda^n e_n \right) = \sum_{n=1}^{\infty} \lambda^n e_{n-1}$$

$$= \sum_{n=0}^{\infty} \lambda^{n+1} e_n = \lambda \sum_{n=0}^{\infty} \lambda^n e_n.$$

It follows that λ is an eigenvalue of S^* and so $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ is contained in the spectrum $\mathrm{sp}(S^*)$ of S^*.

Since $\|S^*\| \leq 1$ we have $\mathrm{sp}(S^*) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. We obtain $\mathrm{sp}(S^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Hence $\mathrm{sp}(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. □

Note that S has no eigenvalues. In fact, if λ is an eigenvalue of S, then λ is different from 0 since S is $1-1$. Let $\sum_{n=0}^{\infty} a_n e_n$ be an eigenvector for λ. We have

$$S \left(\sum_{n=0}^{\infty} a_n e_n \right) = \lambda \left(\sum_{n=0}^{\infty} a_n e_n \right) \iff \sum_{n=0}^{\infty} a_n e_{n+1} = \sum_{n=0}^{\infty} \lambda a_n e_n$$

$$\iff \sum_{n=1}^{\infty} a_{n-1} e_n = \sum_{n=0}^{\infty} \lambda a_n e_n.$$

It follows that $a_0 = 0$ and $a_{n-1} = \lambda a_n$ for $n = 1, 2, \ldots$. Hence $a_n = 0$ for all n.

3. Invariant subspaces of the shift. Let H be a Hilbert space. We will use the symbol $\perp$ to denote orthogonality between vectors, subspaces of H or between a vector and a subspace of H. If W is a closed subspace of a Hilbert space H, we will denote by $W^\perp$ the orthogonal complement of W.

Let $T \in \mathcal{B}(H)$. A subspace W of H is *invariant* by T (or T-*invariant*) if $Tx \in W$ for all $x \in W$. A subspace W of H *reduces* T if W and $W^\perp$ are invariant by T. In this case we say that W is *reducing* for T.

Theorem 3.1. *The shift* S *has no reducing subspaces.*

P r o o f. Assume that W is a reducing subspace for S. Let $S|_W$ (resp. $S|_{W^\perp}$) be the restriction of S to W (resp. to $W^\perp$). Since S is $1-1$ so are $S|_W$

and $S|_{W^\perp}$. Since the range of S has codimension 1, one of $S|_W$ and $S|_{W^\perp}$, say $S|_W$ is onto and hence invertible. Then its inverse is $(S|_W)^*$ and it is an isometry. Now take x in W. We have $\|(S^*)^n x\| = \|((S|_W)^*)^n x\| = \|x\|$. But $(S^*)^n x \to 0$, a contradiction. $\square$

In what follows we will need another representation of the shift operator.

Let dx be the Lebesgue measure on $[0, 2\pi)$. We denote by dm the measure $\frac{dx}{2\pi}$. We consider the space $L^p(\mathbb{T})$ for $p = 1, 2$ with respect to the measure dm. That is, the space of equivalence classes of measurable functions $f : \mathbb{T} \to \mathbb{C}$ which satisfy:

$$\|f\|_p^p = \int_0^{2\pi} |f(e^{ix})|^p dm(x) < +\infty$$

For $f \in L^1(\mathbb{T})$ define

$$\hat{f}(n) = \int_0^{2\pi} f(e^{ix}) e^{-inx} dm(x), \qquad n \in \mathbb{Z}.$$

The map

$$\mathcal{F} : f \to (\hat{f}(n))_{n\in\mathbb{Z}}$$

is the Fourier transform.

The following theorem is well known [4, Theorem 2.7].

Theorem 3.2. *If $f \in L^1(\mathbb{T})$ satisfies $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f = 0$ a.e..*

Note that $L^2(\mathbb{T})$ is a Hilbert space for the scalar product

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{ix})\overline{g(e^{ix})} dm(x)$$

and the family

$$\{\zeta_n : n \in \mathbb{Z}\} \quad \text{where } \zeta_n(e^{ix}) = e^{inx}$$

is orthonormal: $\langle \zeta_n, \zeta_m\rangle = \delta_{nm}$.

Theorem 3.2 shows that no nonzero element of $L^2(\mathbb{T})$ can be orthogonal to the family $\{\zeta_n : n \in \mathbb{Z}\}$: hence it must be an orthonormal basis of $L^2(\mathbb{T})$.

Therefore for each $f \in L^2(\mathbb{T})$ we have

$$f = \sum_{n=-\infty}^{+\infty} \hat{f}(n)\zeta_n \qquad (L^2(\mathbb{T}) \text{ convergence})$$

$$\text{and } \|f\|_2^2 = \sum_{n=-\infty}^{+\infty} |\hat{f}(n)|^2 \qquad \text{(Parseval)}.$$

The *Hardy space* $H^2(\mathbb{T})$ is defined by:

$$H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \hat{f}(-k) = 0 \text{ for all } k = 1, 2, \ldots\}$$

and is a closed subspace of the Hilbert space $L^2(\mathbb{T})$.

We denote the function ζ_1 by ζ.

Let $T_\zeta : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the operator defined by

$$T_\zeta f = \zeta f = \zeta_1 f \qquad (f \in L^2(\mathbb{T})).$$

The Fourier trasform $\mathcal{F}$ is an isomorphism from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$ [4, Theorem 5.5]. Let $\{e_n\}_{n\in Z}$ be the orthonormal basis of $\ell^2(\mathbb{Z})$ defined by

$$e_n(m) = \delta_{nm}.$$

Clearly $\mathcal{F}\zeta_n = e_n$. Let H be the closed subspace of $\ell^2(\mathbb{Z})$ generated by the vectors $e_n : n = 0, 1, 2, \ldots$. Denote by $\mathcal{F}_0 : H^2(\mathbb{T}) \to H$ the operator defined by $\mathcal{F}_0\zeta_n = \mathcal{F}\zeta_n = e_n$. Then, $\mathcal{F}_0$ is a unitary operator and if S is the shift operator on H with respect to the basis $e_n : n = 0, 1, 2, \ldots$ we have:

$$S\mathcal{F}_0 = \mathcal{F}_0 T_\zeta.$$

From this equality and Theorem 2.1 we obtain:

Theorem 3.3. *The operator T_ζ is a non-unitary isometry. Moreover,*

$$\bigcap_{n\geq 0} T_\zeta^n(H^2(\mathbb{T})) = \{0\}.$$

A function $\phi \in H^2(\mathbb{T})$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ is called an *inner function*. Examples are: ζ^n $(n \in \mathbb{N})$. Note that since $|\phi| = 1$ a.e., ϕ defines a bounded, in fact an isometric operator T_ϕ on $H^2(\mathbb{T})$ by the formula

$$T_\phi f = \phi f, \qquad f \in H^2(\mathbb{T}).$$

Therefore the set

$$\phi H^2(\mathbb{T}) = \{\phi f : f \in H^2(\mathbb{T})\}$$

is a closed subspace of $H^2(\mathbb{T})$ since T_ϕ is isometric.

Also, $\phi H^2(\mathbb{T})$ is T_ζ-invariant because $\zeta H^2(\mathbb{T}) \subseteq H^2(\mathbb{T})$ and so

$$T_\zeta(\phi H^2(\mathbb{T})) = \zeta\phi H^2(\mathbb{T}) = \phi\zeta H^2(\mathbb{T}) \subseteq \phi H^2(\mathbb{T}).$$

Moreover,

$$\bigcap_{n\geq 0} T_\zeta^n(\phi H^2(\mathbb{T})) \subseteq \bigcap_{n\geq 0} T_\zeta^n(H^2(\mathbb{T})) = \{0\}.$$

The following theorem shows that every invariant subspace of the operator T_ζ is of the form $\phi H^2(\mathbb{T})$ where ϕ is an inner function.

Theorem 3.4 (Beurling)**.** *A closed nonzero subspace $E \subseteq H^2(\mathbb{T})$ is T_ζ-invariant if and only if there exists $\phi \in H^2(\mathbb{T})$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ such that $E = \phi H^2(\mathbb{T})$. Moreover, ϕ is essentially unique in the sence that if $E = \psi H^2(\mathbb{T})$ where $|\psi| = 1$ a.e. then $\dfrac{\phi}{\psi}$ is a.e. equal to a constant of modulus* 1.

P r o o f. Suppose that $E \subseteq H^2(\mathbb{T})$ is a closed nonzero T_ζ-invariant subspace. The space $T_\zeta(E)$ is a closed subspace of E because T_ζ is isometric.

Moreover, $T_\zeta(E) \subsetneq E$ because

$$\bigcap_{n\geq 0} T_\zeta^n(E) \subseteq \bigcap_{n\geq 0} T_\zeta^n(H^2(\mathbb{T})) = \{0\}.$$

Thus there exists $\phi \in E$ of norm 1*, such that $\phi \perp T_\zeta(E)$.*

Claim 1. *The sequence $\{\phi, T_\zeta(\phi), T_\zeta^2(\phi), \dots\}$ is an orthonormal sequence in E.*

P r o o f. Since $\phi \in E$ which is T_ζ-invariant we have $T_\zeta^n(\phi) \in E$ for all $n \in \mathbb{N}$. Moreover $\|T_\zeta^n(\phi)\|_2 = \|\phi\|_2 = 1$. Let $m, n \in \mathbb{N}$ with $m > n$. We have $\phi \perp T_\zeta(E)$ by construction and so $T_\zeta^n(\phi) \perp T_\zeta^n(T_\zeta(E))$ since T_ζ^n is isometric. On the other hand

$$T_\zeta^m(\phi) \in T_\zeta^m(E) \subseteq T_\zeta^{n+1}(E) = T_\zeta^n(T_\zeta(E)).$$

Thus

$$T_\zeta^n(\phi) \perp T_\zeta^m(\phi).$$

Claim 2. *For all nonzero $k \in \mathbb{Z}$ we have $\displaystyle\int \zeta_k|\phi|^2 dm = 0$.*

P r o o f. For $k > 0$ we have

$$\int \zeta_k|\phi|^2 dm = \int (\zeta_k\phi)\bar{\phi} dm = \langle \zeta_k\phi, \phi\rangle = \left\langle T_\zeta^k(\phi), \phi \right\rangle = 0$$

by the previous claim. For $k = -n < 0$,

$$\int \zeta_k |\phi|^2 dm = \int \phi(\overline{\zeta_n \phi}) dm = \langle \phi, \zeta_n \phi \rangle = \left\langle \phi, T_\zeta^n(\phi) \right\rangle = 0.$$

It follows from this claim that the function $\psi = |\phi|^2$, which is in $L^1(\mathbb{T})$, satisfies $\hat{\psi}(k) = 0$ for all $k \in \mathbb{Z}$ except for $k = 0$. By Theorem 3.2, ψ must be a complex multiple of ζ_0 and hence a.e. equal to a constant. Hence so is $|\phi|$. Since $\int |\phi|^2 dm = 1$, the constant must be 1.

This shows that $|\phi(z)| = 1$ a.e.

Claim 3. $E = \phi H^2(\mathbb{T})$.

P r o o f. Since $\{\zeta_0, \zeta_1, \zeta_2, \cdots\}$ is an orthonormal basis of $H^2(\mathbb{T})$ and T_ϕ is an isometry, the set

$$\{T_\phi \zeta_0, T_\phi \zeta_1, T_\phi \zeta_2, \dots\} = \{\phi, \zeta_1 \phi, \zeta_2 \phi, \dots\} = \{\phi, T_\zeta(\phi), T_\zeta^2(\phi), \dots\}$$

is an orthonormal basis of $\phi H^2(\mathbb{T})$, and is contained in E since $\phi \in E$ which is T_ζ-invariant. We conclude that $\phi H^2(\mathbb{T}) \subseteq E$.

To prove that equality in fact holds, suppose $f \in E$ is orthogonal to $\phi H^2(\mathbb{T})$; we show that $f = 0$. Indeed, for all $n = 0, 1, 2, \dots$ we have

$$f \perp \phi \zeta_n \quad \Rightarrow \quad \int f \overline{\phi \zeta_n} dm = 0 \quad \Rightarrow \quad \int f \bar{\phi} \zeta_{-n} dm = 0.$$

On the other hand if $k = 1, 2, \dots$ then $\zeta_k f = T_\zeta^k(f) \in T_\zeta^k(E) \subseteq T_\zeta(E)$ while $\phi \perp T_\zeta(E)$ by definition; thus $\langle \zeta_k f, \phi \rangle = 0$ and hence

$$\int \zeta_k f \bar{\phi} dm = \langle \zeta_k f, \phi \rangle = 0.$$

The function $f\bar{\phi}$ lies in $L^2(\mathbb{T})$ and has all its Fourier coefficients equal to 0. Hence $f\bar{\phi}$ must vanish a.e. Since $|\phi| = 1$ a.e. this shows that $f = 0$.

Uniqueness

If $\phi H^2(\mathbb{T}) = \psi H^2(\mathbb{T})$ where $|\phi| = |\psi| = 1$ a.e. then $\overline{\psi} \phi H^2(\mathbb{T}) = H^2(\mathbb{T})$, so that $\overline{\psi}\phi = \overline{\psi}\phi\zeta_0 \in H^2(\mathbb{T})$. Similarly $\overline{\phi}\psi H^2(\mathbb{T}) = H^2(\mathbb{T})$, so that $\overline{\phi}\psi \in H^2(\mathbb{T})$. Thus the function $h = \overline{\psi}\phi$ and its complex conjugate are both analytic, which can only happen if h is a constant.

This concludes the proof of the Theorem. □

4. The C^*-algebra generated by the shift. If H is Hilbert space we will denote by $\mathcal{K}(H)$ the algebra of compact operators on H. If $x \in H$, $y \in H$ we will denote by $x \otimes y$ the rank-one operator on H defined by $x \otimes y(z) = \langle z, x\rangle\, y$.

A C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is *irreducible* if there is no closed subspace of H invariant for all operators in $\mathcal{A}$, other than H and $\{0\}$.

An important property of an irreducible C^*-algebra is the following [3, Corollary I.10.4]:

Theorem 4.1. *An irreducible C^*-subalgebra of $\mathcal{B}(H)$ that contains a nonzero compact operator contains $\mathcal{K}(H)$.*

Theorem 4.2. *Let π be an irreducible representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space H. If $\mathcal{J}$ is an ideal of $\mathcal{A}$ and $\pi(\mathcal{J}) \neq \{0\}$, then the restriction $\pi|\mathcal{J}$ of π to $\mathcal{J}$ is irreducible.*

P r o o f. Let $V = \{x \in H : \ \pi(\mathcal{J})x = \{0\}\}$. Then V is invariant by π and since π is irreducible, $V = \{0\}$. So if $x \in H$, $x \neq 0$, we have $\pi(\mathcal{J})x \neq \{0\}$. But, $\pi(\mathcal{J})x$ is invariant under π and so is $\overline{\pi(\mathcal{J})x}$. Hence $\overline{\pi(\mathcal{J})x}$ equals H. This proves that every non-zero vector of H is cyclic for $\pi|\mathcal{J}$ and hence $\pi|\mathcal{J}$ is irreducible. □

We will denote by $L^\infty(\mathbb{T})$ the space of equivalence classes of essentially bounded functions $f : \mathbb{T} \to \mathbb{C}$ with respect to the measure dm, equipped with the supremum norm.

Let g be a function in $L^\infty(\mathbb{T})$. The *multiplication operator* $M_g : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by the formula $M_g h = gh$.

The *Toeplitz operator* $T_g : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is defined by the formula $T_g h = Pgh$, where P is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Proposition 4.3. *For $g \in L^\infty(\mathbb{T})$, $T_g^* = T_{\overline{g}}$, $\|T_g\| = \|T_g\|_e = \|g\|_\infty$.*

P r o o f. We have for $f \in H^2(\mathbb{T})$, $h \in H^2(\mathbb{T})$

$$\left\langle T_g^* f, h\right\rangle = \langle f, Pgh\rangle = \langle f, gh\rangle = \langle \overline{g}h, f\rangle = \left\langle T_{\overline{g}} f, h\right\rangle .$$

Also it is clear that

$$\|T_g\|_e \leq \|T_g\| \leq \|M_g\| = \|g\|_\infty .$$

We show that $\|g\|_\infty \leq \|T_g\|_e$.

Let $\epsilon > 0$. Since the trigonometric polynomials are dense in $L^2(\mathbb{T})$, there exists a polynomial $p = \sum_{k=-N}^{N} a_k \zeta_k$ with $\|p\|_2 = 1$ such that $\|gp\|_2 > \|g\|_\infty - \epsilon$.

For all $n > N$ the function $\zeta_n p$ belongs to $H^2(\mathbb{T})$. Set

$$gp = \sum_{k=-\infty}^{\infty} b_k \zeta_k.$$

Then

$$T_g \zeta_n p = P\zeta_n gp = \sum_{k=-n}^{\infty} b_k \zeta_{k+n}$$

and hence

$$\lim_{n\to\infty} \|T_g \zeta_n p\|_2 = \|gp\|_2.$$

The sequence $\zeta_n p$ converges weakly to 0. Hence, if K is a compact operator, the sequence $K\zeta_n p$ converges to 0 and we have

$$\lim_{n\to\infty} \|(T_g + K)\zeta_n p\|_2 = \lim_{n\to\infty} \|T_g \zeta_n p\|_2 = \|gp\|_2.$$

On the other hand, for all $n > N$

$$\|(T_g + K)\zeta_n p\|_2 \leq \|T_g + K\| \|M_{\zeta_n} p\|_2 \leq \|T_g + K\| \|M_{\zeta_n}\| \|p\|_2 \leq \|T_g + K\|.$$

It follows that

$$\|gp\|_2 \leq \|T_g + K\|$$

and so

$$\|gp\|_2 \leq \|T_g\|_e.$$

We conclude that

$$\|g\|_\infty - \epsilon < \|T_g\|_e$$

and hence

$$\|g\|_\infty \leq \|T_g\|_e. \qquad \square$$

The *Hardy space* $H^\infty(\mathbb{T})$ is defined by:

$$H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : \hat{f}(-k) = 0 \text{ for all } k = 1, 2, \ldots\}.$$

Proposition 4.4. *For $h \in H^\infty(\mathbb{T})$ the space $H^2(\mathbb{T})$ is invariant by M_h. If $g \in L^\infty(\mathbb{T})$ and $h \in H^\infty(\mathbb{T})$ then*

$$T_g T_h = T_{gh}$$

$$T_{\overline{h}} T_g = T_{\overline{h}g}.$$

P r o o f. We have $h = \sum_{n=0}^{\infty} h_n \zeta_n$. Hence if $k = 0, 1, 2, \dots$ we have

$$M_h \zeta_k = h\zeta_k = M_{\zeta_k} h = \sum_{n=0}^{\infty} h_n \zeta_{n+k}$$

which is in $H^2(\mathbb{T})$. Hence, the space $H^2(\mathbb{T})$ is invariant by M_h.

Now, if $f \in H^2(\mathbb{T})$ we have $T_g T_h f = T_g P h f = T_g h f$ since $hf \in H^2(\mathbb{T})$. Hence $T_g T_h f = T_g h f = P g h f = T_{gh} f$.

The other equality follows by taking adjoints. □

Recall that we denote by ζ the function ζ_1.

Proposition 4.5. *If $g \in L^\infty(\mathbb{T})$ the operator*

$$T_g T_\zeta - T_\zeta T_g$$

has rank at most one.

P r o o f. By the previous proposition we have $T_{g\zeta} = T_g T_\zeta$. So,

$$T_\zeta T_g - T_{g\zeta} = (P M_\zeta P M_g P - P M_\zeta M_g P)|H^2(\mathbb{T}) = (P M_\zeta P^\perp M_g P)|H^2(\mathbb{T}).$$

But $P M_\zeta P^\perp$ is the rank one operator $\zeta_{-1} \otimes \zeta_0$ and hence the commutator has rank at most one. □

We denote by $C(\mathbb{T})$ the space of continuous functions $f : \mathbb{T} \to \mathbb{C}$.

Proposition 4.6. *For $g \in L^\infty(\mathbb{T})$ and $f \in C(\mathbb{T})$, the operators $T_f T_g - T_{fg}$ and $T_g T_f - T_{gf}$ are compact.*

P r o o f. Let $\epsilon > 0$ and $p(z) = \sum_{k=-N}^{N} a_k \zeta_k$ be a trigonometric polynomial such that $\|f - p\|_\infty < \epsilon$. As in the previous proposition we obtain

$$T_g T_f - T_{gf} = P M_g P^\perp M_f P | H^2(\mathbb{T})$$

It suffices to show that $P^\perp M_f P$ is compact. The range of $P^\perp M_p P$ is contained in the span of the set $\{\zeta_{-k} : 1 \le k \le N\}$ and so it is a finite rank operator. Since $\|P^\perp (M_f - M_p) P\| \le \|f - p\|_\infty < \epsilon$, the operator $P^\perp M_f P$ is a limit of finite rank operators and hence it is compact.

The other assertion follows by taking adjoints. □

Theorem 4.7. *Let $C^*(T_\zeta)$ be the C^*-algebra generated by T_ζ. Let*

$$\mathcal{T}(C(\mathbb{T})) = \{T_f + K : f \in C(\mathbb{T}), K \in \mathcal{K}(H^2(\mathbb{T}))\}.$$

Then

(1) *The C^*-algebra $C^*(T_\zeta)$ is equal to $\mathcal{T}(C(\mathbb{T}))$.*
(2) *The algebra $C^*(T_\zeta)$ is irreducible and contains $\mathcal{K}(H^2(\mathbb{T}))$ as its unique minimal ideal.*
(3) *We have the following exact sequence:*

$$0 \to \mathcal{K}(H^2(\mathbb{T})) \to C^*(T_\zeta) \underset{s}{\overset{\pi}{\rightleftarrows}} C(\mathbb{T}) \to 0$$

where π be the quotient map of $C^*(T_\zeta)$ onto $C^*(T_\zeta)/\mathcal{K}(H^2(\mathbb{T}))$ and $s : C(\mathbb{T}) \to C^*(T_\zeta)$ is the map defined by $s(f) = T_f$.
(4) *The map s is a continuous section of the exact sequence.*

P r o o f. Let Q be a projection commuting with T_ζ. Then Q commutes with $I - T_\zeta T_\zeta^*$ which is the rank-one operator $\zeta_0 \otimes \zeta_0$. And so, $Q\zeta_0 = \zeta_0$ or $Q\zeta_0 = 0$. If $Q\zeta_0 = \zeta_0$, then $Q\zeta_n = QT_\zeta^n\zeta_0 = T_\zeta^n Q\zeta_0 = \zeta_n$ for all $n = 0, 1, \ldots$ and hence $Q = I$. If $Q\zeta_0 = 0$, similarly we see that $Q = 0$. Hence the C^*-algebra $C^*(T_\zeta)$ is irreducible.

The C^*-algebra $C^*(T_\zeta)$ is irreducible and contains a compact operator. It follows from Theorem 4.1 that it contains $\mathcal{K}(H)$. On the other hand, since $T_{\zeta^{-1}} = T_\zeta^*$ the algebra $C^*(T_\zeta)$ contains T_p for every trigonometric polynomial p. The trigonometric polynomials are dense in $C(\mathbb{T})$ and the map $f \to T_f$ is continuous (Proposition 4.3). It follows that $C^*(T_\zeta)$ contains T_f for every $f \in C(\mathbb{T})$. So, $C^*(T_\zeta)$ contains $\mathcal{T}(C(\mathbb{T}))$.

Let $\mathcal{J}$ be a non-zero ideal of $C^*(T_\zeta)$. By Theorem 4.2 it acts irreducibly. If $X \in \mathcal{J}$ and $XF = 0$ for every finite-rank operator F, then $X = 0$. It follows that $XF \neq 0$ for some finite-rank operator F and the ideal $\mathcal{J}$ contains compact operators. Since $\mathcal{J}$ contains compact operators and acts irreducibly, it follows from Theorem 4.1 that $\mathcal{J}$ contains all compact operators. Therefore $\mathcal{K}(H^2(\mathbb{T}))$ is the unique minimal ideal of $C^*(T_\zeta)$.

It follows from Proposition 4.6 that the product of Toeplitz operators with continuous symbol is a Toeplitz operator plus a compact operator. Thus $\mathcal{T}(C(\mathbb{T}))$ is a $*$-algebra. We show that it is norm-closed. Suppose that $T_{f_n} + K_n$ converges to an operator X, where f_n is in $C(\mathbb{T})$ and K_n is compact. Since

$$\|f_n - f_m\|_\infty = \|T_{f_n} - T_{f_n}\|_e \leq \|T_{f_n} + K_n - (T_{f_n} + K_m)\|$$

the sequence f_n is a Cauchy sequence and converges to some limit $f \in C(\mathbb{T})$. It follows that K_n is also a Cauchy sequence and converges to some compact operator K. Hence $X = T_f + K$. And so $\mathcal{T}(C(\mathbb{T}))$ is a C^*-algebra. Since it contains T_ζ and is contained in $C^*(T_\zeta)$, we have $C^*(T_\zeta) = \mathcal{T}(C(\mathbb{T}))$.

Let π be the quotient map of $C^*(T_\zeta)$ onto $C^*(T_\zeta)/\mathcal{K}(H^2(\mathbb{T}))$. By Proposition 4.6 this quotient algebra is abelian. The map πs is surjective and by Proposition 4.3 it is isometric. By Proposition 4.6 it is a $*$-homomorphism. Hence it is a $*$-isomorphism. With this identification, s is a continuous section of the quotient map. $\square$

The C^*-algebra generated by the Toeplitz operators T_f with continuous symbol, which by the above Theorem is equal to $\mathcal{T}(C(\mathbb{T}))$, is called the *Toeplitz algebra*.

5. Wold decomposition and Coburn's theorem. In this section we define the shift of arbitrary multiplicity and prove that an isometry is decomposed as a sum of a unitary operator and a shift of some multiplicity. Finally we present Coburn's Theorem which shows that the C^*-algebra generated by a proper isometry is isomorphic to the C^*-algebra generated by the shift.

Let H be a Hilbert space. An operator S acting on H is called a *shift of multiplicity a* if there exists a decomposition $H = \sum_{i=0}^{\infty} \oplus H_i$ of H into a direct sum of mutually orthogonal subspaces H_i, $i = 0, 1, 2, \dots$ of the same dimension a and and S maps H_i isometrically onto H_{i+1}. The common dimension of the H_i's is the *multiplicity* of the shift.

Theorem 5.1 (Wold decomposition)**.** *If $V \in B(H)$ is an isometry, there exists a unique decomposition $H = H_s \oplus H_u$ into V-reducing subspaces such that the restriction V_s of V to H_s is a shift of some multiplicity (if nonzero) and the restriction V_u of V to H_u is unitary (if nonzero).*

Set $L = H \ominus V(H)$. Let $x \in L$, $y \in L$ and $n > m > 0$. We have $\langle V^n x, V^m y\rangle = \langle V^{n-m}x, y\rangle = 0$ since $y \in L$ and $V^{n-m}x \in VH$. Thus the family $\{V^n(L) : n \in \mathbb{Z}_+\}$ is a family of closed mutually orthogonal subspaces.

We set

$$H_s = \sum_{n\geq 0} \oplus V^n(L).$$

Then H_s is invariant by V and since $V(V^n(L)) = V^{n+1}(L)$, V restricted to H_s is a shift of multiplicity $\dim L$.

Observe that $V^n(L) = V^n(H) \ominus V^{n+1}(H)$. Indeed let $w \in V^n(H) \ominus V^{n+1}(H)$. There exists $x \in H$ such that $w = V^n x$. Write $x = y + z$ with $y \in L$, $z \in VH$. Since $V^n z \in V^{n+1}(H)$ we have $\langle V^n y + V^n z, V^n z\rangle = \langle V^n x, V^n z\rangle = \langle w, V^n z\rangle = 0$.

Since V preserves orthogonality, $\langle V^n y, V^n z\rangle = 0$. Hence $\langle V^n z, V^n z\rangle = 0$. It follows that $z = 0$, and so $w = V^n y \in V^n L$.

Set

$$H_u = \bigcap_{n\geq 0} V^n(H).$$

It is clear that H_u is invariant by V. Let $x \in H$. Then x is orthogonal to H_s if and only if x is orthogonal to $V^n(L)$ for every $n \geq 0$, if and only if $x \in V^n(H)$ for every $n \geq 1$, that is if and only $x \in H_u$. Hence $H = H_s \oplus H_u$. Finally it is clear that H_u is invariant by V and V restricted to H_u is surjective. Since it is also isometric it is unitary. □

An isometry which is not a unitary operator is called *proper*.

Theorem 5.2 (Coburn [2])**.** *Let V be a proper isometry and $C^*(V)$ the C^*-algebra generated by V. Then there is a unique $*$-isomorphism ϕ of $C^*(T_\zeta)$ onto $C^*(V)$ such that $\phi(T_\zeta) = V$.*

P r o o f. It follows from Theorem 5.1 that V is unitarily equivalent to $T_\zeta^a \oplus U$ where T_ζ^a is a shift of multiplicity a and U is a unitary operator. We identify V with $T_\zeta^a \oplus U$. Define ϕ from $C^*(T_\zeta)$ into $C^*(V)$ by

$$\phi(T_f + K) = (T_f + K)^a \oplus f(U)$$

for $f \in C(\mathbb{T})$, $K \in \mathcal{K}(H^2(\mathbb{T}))$. The map from $C^*(T_\zeta)$ to $C(\mathbb{T})$ taking $T_f + K$ to f is a $*$-homomorphism and by the normal functional calculus the map taking f to $f(U)$ is a $*$-homomorphism. It follows that the composition map $T_f + K \to f \to f(U)$ is a $*$-homomorphism from $C^*(T_\zeta)$ onto $C^*(U)$. Thus ϕ is a $*$-monomorphism and $\phi(T_\zeta) = V$. As the image of ϕ is a C^*-algebra, it is surjective. □

REFERENCES

[1] A. Beurling. On two problems concerning linear transformations in Hilbert space. *Acta Math., Uppsala* **81** (1949), 239–255.

[2] L. Coburn. C^*-algebra of an isometry. *Bull. Am. Math. Soc.* **73** (1967), 722–726.

[3] K. R. Davidson. C^*-algebras by example. Fields Institute Monographs, vol. **6**. Providence, RI, American Mathematical Society, 1996.

[4] Y. Katznelson. An Introduction to Harmonic Analysis, 3rd ed. Cambridge Mathematical Library. Cambridge, Cambridge University Press, 2004.

M. S. Anoussis
Department of Mathematics
University of the Aegean
83200 Karlovassi (Samos), Greece
e-mail: mano@aegean.gr

Received July 1, 2013