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THE C*-ALGEBRA OF A LOCALLY COMPACT GROUP[*]

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ABSTRACT. In this note, we briefly introduce the C*-algebra of a locally compact group and present some important structural results.

This note is a comprehensive but brief introduction to the basic notions of non-commutative Harmonic Analysis. Our main references for the subject are J. Dixmier's book [2] and G. Folland's book [4].

1. Convolution algebras. A group G is called a *topological group* if it is equipped with a topology such that the group operations are continuous. By a *locally compact group* we mean a topological group whose topology is locally compact and Hausdorff. Examples include the additive group $\mathbb{R}^n$, $\mathbb{Z}$, the group $\mathbb{T}$ of complex numbers of modulus one, and the group $GL(n, \mathbb{R})$ of invertible $n \times n$ matrices over $\mathbb{R}$.

Let G be a locally compact group. We recall that $C_c(G)$ is the space of all continuous functions with compact supports on G, and $C_c^+(G)$ is the space of positive elements in $C_c(G)$. A *left Haar measure* on G is a nonzero Radon measure μ on G which is left invariant, this is, $\mu(xE) = \mu(E)$ for every Borel set $E \subset G$ and $x \in G$. Similarly, we can define the *right Haar measure* on G. We have the following important results for left (resp. right) Haar measures.

Proposition 1.1. *Let μ be a Radon measure on the locally compact group G.*

(i) *μ is a left Haar measure if and only if the measure $\tilde{\mu}$ given by $\tilde{\mu}(E) := \mu(E^{-1})$ is a right Haar measure.*

(ii) *μ is a left Haar measure if and only if $\int L_y f d\mu = \int f d\mu$, for all $y \in G$, $f \in C_c^+(G)$, where $L_y f(x) := f(y^{-1}x)$ is the left translation of the function f on G.*

From Proposition 1.1 (*i*), we see that it is of little importance whether one chooses to study left or right Haar measure. The more common choice is the left one, however. The following existence theorem is of fundamental theoretical importance; but for most specific groups one can actually construct Haar measure in an explicit fashion.

Theorem 1.2. *Every locally compact group G possesses a left Haar measure.*

On the other hand, we have the following uniqueness theorem for left Haar measures.

Theorem 1.3. *If λ and μ are left Haar measures on G, then there exists $c \in (0, \infty)$ such that $\mu = c\lambda$.*

Examples of left Haar measures for topological groups mentioned above are the Lebesgue measure dx on $\mathbb{R}$; counting measure δ on $\mathbb{Z}$ and $|\det T|^{-n} dT$ on the group $GL(n, \mathbb{R})$, where dT is the Lebesgue measure on the vector space of all real $n \times n$ matrices. The *ax + b-group* is the group of all affine transformations $x \mapsto ax + b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$, and $da\,db/a^2$ is a left Haar measure.

Let G be a locally compact group with left Haar measure λ. If for $x \in G$, we define $\lambda_x(E) := \lambda(Ex)$, then by the associative law $y(Ex) = (yE)x$, we have

that λ_x is again a left Haar measure. By the uniqueness theorem above (Theorem 1.3), there is a positive number $\Delta(x)$ which depends on x but is independent of λ such that $\lambda_x = \Delta(x)\lambda$. The function $\Delta : G \to (0, \infty)$ thus defined is called the *modular function* of G.

Proposition 1.4.

(i) Δ *is a continuous homomorphism from G to the multiplicative group of positive real numbers.*

(ii) *For any $f \in L^1(\lambda)$,* $\int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda$, *where* $R_y f(x) := f(xy)$.

(iii) *For every left Haar measure λ, the associated right Haar measure ρ satisfies*

$$d\rho(x) = \Delta(x^{-1})d\lambda(x).$$

We see that if we set $y_0 = y^{-1}$ in Proposition 1.4 (ii) and make the substitution $x \to xy_0$, we have

$$\Delta(y_0) \int f(x) d\lambda(x) = \int f(xy_0^{-1}) d\lambda(x) = \int f(x) d\lambda(xy_0),$$

which gives a convenient abbreviated form of Proposition 1.4 (ii)

$$d\lambda(xy_0) = \Delta(y_0) d\lambda(x).$$

On the other hand, G is called *unimodular* if $\Delta \equiv 1$, that is, the left Haar measure is also a right Haar measure. Obviously abelian groups and discrete groups are unimodular, but many other groups are too. Here we give some classes of examples .

Proposition 1.5. *If K is any compact subgroup of G, then $\Delta|_K \equiv 1$.*

P r o o f. We have that $\Delta(K)$ is a compact subgroup of $\mathbb{R}^+$, the multiplicative group of positive real numbers, hence $\Delta(K) = \{1\}$. $\square$

Corollary 1.6. *If G is compact, then G is unimodular.*

Proposition 1.7. *If $G/[G, G]$ is compact, then G is unimodular.*

As a consequence of Proposition 1.7, we see that every connected semisimple Lie group is unimodular.

From now on we shall assume that each locally compact group G is equipped with a fixed left Haar measure λ. We shall generally write dx for $d\lambda(x)$, $\int f$ for $\int f d\lambda$, and $L^p(G)$ for $L^p(\lambda)$.

Let G be a locally compact group. For f and g in $L^1(G)$, the convolution of f and g is defined by

$$f * g(x) := \int_G f(y)g(y^{-1}x)dy, \quad \text{for} \quad x \in G.$$

It follows from Fubini's theorem that the integral is absolutely convergent for almost every x and that $\|f * g\|_1 \leq \|f\|_1\|g\|_1$. Moreover, the involution defined on $L^1(G)$ is given by

$$f^*(x) := \Delta(x^{-1})\overline{f(x^{-1})} \quad \text{for} \quad f \in L^1(G), \ x \in G.$$

With the convolution product and the involution, $L^1(G)$ is a Banach $*$-algebra, called the L^1-*group algebra* of G. Note that $f * g(x)$ can be expressed in several different forms:

$$\begin{aligned}
(1) \qquad f * g(x) &= \int_G f(y)g(y^{-1}x)dy \\
&= \int_G f(xy)g(y^{-1})dy \\
&= \int_G f(y^{-1})g(yx)\Delta(y^{-1})dy \\
&= \int_G f(xy^{-1})g(y)\Delta(y^{-1})dy,
\end{aligned}$$

that is, $f * g = \int_G f(y)L_y g \, dy = \int_G g(y^{-1})R_y f \, dy$. Thus, $f * g$ is a generalized linear combination of left translations of g, or of right translations of f. Since $L_x R_y = R_y L_x$, we have

$$L_z(f * g) = (L_z f) * g \quad \text{and} \quad R_z(f * g) = f * (R_z g).$$

Moreover, convolution can be extended from L^1 to other L^p spaces. We give the following results for the ideas.

Proposition 1.8. *Suppose $1 \leq p \leq \infty$, $f \in L^1$ and $g \in L^p$.*

(i) *The integrals in (1) converge absolutely for almost every x and we have $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.*

(ii) *When $p = \infty$, $f * g$ is continuous.*

Proposition 1.9. *Suppose G is unimodular. If $f \in L^p(G)$ and $g \in L^q(G)$ where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $f * g \in C_0(G)$ and $\|f * g\|_{\sup} \leq \|f\|_p \|g\|_q$.*

2. Unitary representations. In this section, we will talk about unitary representations of a topological group and their basic theory.

Let G be a locally compact group. We say that π is a *unitary representation* of G on the Hilbert space $\mathcal{H}_\pi$ if $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a homomorphism and continuous in the strong operator topology, where $\mathcal{U}(\mathcal{H}_\pi)$ is the group of unitary operators on some nonzero Hilbert space $\mathcal{H}_\pi$. The dimension of $\mathcal{H}_\pi$ is called the *dimension* or *degree* of π. A basic example is given when G is a locally compact group with a left Haar measure. Then left translations yield a unitary representation π_L of G on $L^2(G)$ called the *left regular representation*, by taking $(\pi_L(x)f)(y) := L_x f(y) = f(x^{-1}y)$. Similarly, one can define the *right regular representation* π_R of G on $L^2(G, \rho)$, where ρ is a right Haar measure on G by

$$[\pi_R(x)f](y) = R_x(f)(y) = f(yx).$$

On the other hand, one can also define a unitary representation $\widetilde{\pi}_R$ on $L^2(G)$ (with left Haar measure) by:

$$[\widetilde{\pi}_R(x)f](y) = \Delta(x)^{1/2} R_x(f)(y) = \Delta(x)^{1/2} f(yx).$$

Here both π_R and $\widetilde{\pi}_R$ are called the *right regular representation* of G.

We introduce some standard terminology associated to unitary representations. If π_1, π_2 are two unitary representations of G, an *intertwining operator* for π_1 and π_2 is a bounded linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. Denote by $\mathcal{C}(\pi_1, \pi_2)$ the set of all such operators. We say that π_1 and π_2 are *unitary equivalent* if $\mathcal{C}(\pi_1, \pi_2)$ contains a unitary operator U; then $\pi_2(x) = U\pi_1(x)U^{-1}$. Note that the right regular representation π_R is unitary equivalent to the left regular representation π_L by taking $Uf(x) = f(x^{-1})$; and

the right regular representations π_R on $L^2(G, \rho)$ and $\widetilde{\pi}_R$ on $L^2(G, \lambda)$ are equivalent by taking $Uf = \Delta^{1/2} f$.

Suppose $\mathcal{M}$ is a closed subspace of $\mathcal{H}_\pi$; it is called an *invariant subspace* for π if $\pi(x)\mathcal{M} \subset \mathcal{M}$ for all $x \in G$. If π admits a nontrivial invariant subspace, then π is called *reducible*, otherwise, π is *irreducible*. If π is a unitary representation of G and $u \in \mathcal{H}_\pi$, the closed linear span of $\{\pi(x)u : x \in G\}$ in $\mathcal{H}_\pi$ denoted by $\mathcal{M}_u$ is called the *cyclic subspace* generated by u. It is clear that $\mathcal{M}_u$ is invariant under π. If $\mathcal{M}_u = \mathcal{H}_\pi$, then u is called a *cyclic vector* for π; and if π has a cyclic vector, then it is called a *cyclic representation*. If $\{\pi_i\}_{i \in I}$ is a family of unitary representations, their *direct sum* $\oplus\pi_i$ is the representation π on $\mathcal{H} = \oplus\mathcal{H}_{\pi_i}$ defined by $\pi(x)(\sum v_i) := \sum \pi_i(x)(v_i)$, where $v_i \in \mathcal{H}_{\pi_i}$.

Proposition 2.1. *Every unitary representation is a direct sum of cyclic representations.*

Lemma 2.2 (Schur's Lemma)**.**

(i) *A unitary representation π of G is irreducible if and only if $\mathcal{C}(\pi)(:= \mathcal{C}(\pi, \pi))$ contains only scalar multiples of the identity.*

(ii) *Suppose π_1 and π_2 are irreducible unitary representations of G. If they are equivalent, then $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional; otherwise, $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.*

As a corollary, we see that if G is abelian, then every irreducible representation of G is one-dimensional. This will be used for an example later on.

Let G be a locally compact group; we saw in the previous section that $L^1(G)$ is a Banach $*$-algebra under convolution product and the involution. We are going to see that there is actually a one-to-one correspondence between the unitary representations of G and the nondegenerate $*$-representations of $L^1(G)$. Let $\mathcal{A}$ be an abstract Banach $*$-algebra. A $*$-*representation* of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism π from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. We see that the norm closure of $\pi(\mathcal{A})$ in $\mathcal{B}(\mathcal{H})$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$, and we say that π is *nondegenerate* if there is no nonzero $v \in \mathcal{H}$ such that $\pi(x)v = 0$ for all $x \in \mathcal{A}$.

Each unitary representation π of G determines a representation (also denoted by π) of $L^1(G)$ in the sense that for $f \in L^1(G)$, a bounded linear operator $\pi(f)$ on $\mathcal{H}_\pi$ is defined by

$$\pi(f) = \int_G f(x)\pi(x)dx.$$

More precisely, for $u, v \in \mathcal{H}_\pi$ we define

$$\langle \pi(f)u, v \rangle := \int_G f(x)\langle \pi(x)u, v \rangle dx.$$

Since $\langle \pi(x)u, v \rangle \in C_b(G)$, $\langle \pi(f)u, v \rangle$ is well-defined. Because $\langle \pi(f)u, v \rangle$ is linear on u, conjugate linear on v and

$$|\langle \pi(f)u, v \rangle| \leq \|f\|_1 \|u\| \|v\|,$$

we have that $\pi(f) \in \mathcal{B}(\mathcal{H}_\pi)$, the space of bounded linear operators on $\mathcal{H}_\pi$, with $\|\pi(f)\| \leq \|f\|_1$. An example can be given through the left regular representation π_L of G, that is, $\pi_L(x) = L_x$ for all x in G. For f in $L^1(G)$, $\pi_L(f)$ is given by

$$\pi_L(f)g = \int_G f(y)L_y g \, dy = f * g.$$

We have the following two theorems to indicate the relation between the unitary representation of G with the nondegenerate $*$-representation of $L^1(G)$.

Theorem 2.3. *If π is a unitary representation of G, then $f \mapsto \pi(f)$ is a nondegenerate $*$-representation of $L^1(G)$ on $\mathcal{H}_\pi$. More precisely, π is a $*$-homomorphism and there is no nonzero $u \in \mathcal{H}_\pi$ such that $\pi(f)u = 0$ for all $f \in L^1(G)$. Moreover, for $x \in G$ and $f \in L^1(G)$,*

$$\pi(x)\pi(f) = \pi(L_x f) \quad and \quad \pi(f)\pi(x) = \Delta(x^{-1})\pi(R_{x^{-1}}f).$$

Theorem 2.4. *Let π be a nondegenerate $*$-representation of $L^1(G)$ on $\mathcal{H}$. Then π arises from a unique unitary representation of G on $\mathcal{H}$ according to the formula*

$$\langle \pi(f)u, v \rangle := \int_G f(x)\langle \pi(x)u, v \rangle dx.$$

Let π be a unitary representation of G. If G is a discrete group, the associated representation of $L^1(G)$ includes the representation of G, since $\pi(x) = \pi(\delta_x)$, where δ_x is the point evaluation at x. But, if G is not discrete and π is infinite-dimensional, the families

$$\pi(G) = \{\pi(x) : x \in G\} \quad and \quad \pi(L^1(G)) = \{\pi(f) : f \in L^1(G)\}$$

are quite different. Indeed, the C*-algebras generated by these two families frequently have trivial intersection. We give the following theorem for information.

Theorem 2.5. *Let π be a unitary representation of G.*

(i)   *The C*-algebras generated by $\pi(G)$ and $\pi(L^1(G))$ have the same closure in the strong and weak operator topology.*

(ii) *The operator $T \in \mathcal{B}(\mathcal{H}_\pi)$ is a intertwining operator for π if and only if $T\pi(f) = \pi(f)T$ for every $f \in L^1(G)$.*

(iii) *A closed subspace $\mathcal{M}$ of $\mathcal{H}_\pi$ is invariant under π if and only if $\pi(f)\mathcal{M} \subset \mathcal{M}$ for every $f \in L^1(G)$.*

3. The group C*-algebra. Let G be a locally compact group and π be a irreducible unitary representation of G on $\mathcal{H}_\pi$. We denote by $[\pi]$ the equivalence class of π, and denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G. This set is called the (unitary) *dual space* of G.

If G is abelian, then every irreducible representation of G is one-dimensional by Schur's lemma. Thus, for each such representation π, we can write $\mathcal{H}_\pi = \mathbb{C}$, and then $\pi(x)(z) = \xi(x)z$ for all $z \in \mathbb{C}$, where ξ is a character of G, that is, a continuous group homomorphism from G into $\mathbb{T}$. Then the dual space $\widehat{G}$ can be identified with the spectrum of $L^1(G)$ via

$$\xi(f) = \int \xi(x)f(x)dx.$$

It is easy to see that $\widehat{G}$ is an abelian group under pointwise multiplication, its identity is the constant function taking value one and

$$\xi^{-1}(x) = \xi(x^{-1}) = \overline{\xi(x)}.$$

Moreover, $\widehat{G}$ can be equipped with a natural topology so that it is locally compact. Therefore, $\widehat{G}$ is a locally compact abelian group, called the *dual group* of G.

We have the following proposition for an abelian group in relation to its dual group.

Proposition 3.1. *If G is discrete then $\widehat{G}$ is compact; if G is compact then $\widehat{G}$ is discrete.*

Let G be a locally compact group. For $f \in L^1(G)$, we define

$$\|f\|_* := \sup_{[\pi] \in \widehat{G}} \|\pi(f)\|.$$

Then $\|\cdot\|_*$ is a norm on $L^1(G)$ such that $\|f\|_* \leq \|f\|_1$. Moreover, we have that

$$\|f * g\|_* = \sup_{[\pi] \in \widehat{G}} \|\pi(f)\pi(g)\| \leq \|f\|_* \|g\|_*,$$

$$\|f^*\|_* = \sup_{[\pi] \in \widehat{G}} \|\pi(f)^*\| = \|f\|_*,$$

$$\|f^* * f\|_* = \sup_{[\pi] \in \widehat{G}} \|\pi(f)^*\pi(f)\| = \sup_{[\pi] \in \widehat{G}} \|\pi(f)\|^2 = \|f\|_*^2.$$

Hence, the convolution and involution on $L^1(G)$ can be extended continuously to the completion of $L^1(G)$ with respect to $\|\cdot\|_*$, which is a C*-algebra called the *group C*-algebra* of G and denoted by $C^*(G)$. We have $\|f\|_* := \sup_{\pi \in \widehat{G}} \|\pi(f)\|$. In the following, we give few examples of group C*-algebras.

Example 3.2. Let $G = \mathbb{Z}$, an abelian discrete group. Suppose that π is an irreducible representation of $\mathbb{Z}$. Then π is one-dimensional, and hence acts on the Hilbert space $\mathbb{C}$. Since $\pi(1)$ is unitary, we have that

$$\pi(1) \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

say $\pi(1) = \zeta$. Then $\pi(n) = \pi(n \cdot 1) = \pi(1)^n = \zeta^n$ for all $n \in \mathbb{Z}$. Conversely, every $\zeta \in \mathbb{T}$ gives rise to an irreducible representation of $\mathbb{Z}$ acting on $\mathbb{C}$ by setting $\pi(n) = \zeta^n$ for $n \in \mathbb{Z}$. Thus, $\widehat{\mathbb{Z}} = \mathbb{T}$.

Now suppose that $\zeta \in \mathbb{T}$, and let π_ζ be the corresponding irreducible representation of $\mathbb{Z}$. Let $f \in \ell^1(\mathbb{Z})$, say $f = (f_n)_{n \in \mathbb{Z}}$. We have that

$$\pi_\zeta(f) = \int_{\mathbb{Z}} f_n \pi_\zeta(n) d\delta_{\mathbb{Z}} = \sum_{n \in \mathbb{Z}} f_n \zeta^n.$$

Let $\hat{f} : \mathbb{T} \to \mathbb{C}$ be the function given by

$$\hat{f}(z) = \sum_{n \in \mathbb{Z}} f_n z^n \quad \text{for} \quad z \in \mathbb{T}.$$

It follows that

$$\|f\|_* = \sup_{\zeta \in \mathbb{T}} \|\pi_\zeta(f)\| = \|\hat{f}\|_\infty.$$

Since the Laurent polynomials are dense in $C(\mathbb{T})$, it follows that $C^*(\mathbb{Z}) \cong C(\mathbb{T})$.

Example 3.3. Let G be an abelian locally compact group. The irreducible representations of G can be identified as in the above example with the characters of G, namely, with the continuous group homomorphisms $\xi : G \to \mathbb{T}$. The set $\widehat{G}$ is a group under pointwise multiplication, and can be equipped with a natural topology which turns it into a locally compact abelian group. Moreover, Pontryagin's theorem identifies the dual of $\widehat{G}$ with G. It can be shown that $C^*(G) \cong C_0(\widehat{G})$. For instance, $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$, since $\widehat{\mathbb{R}} \cong \mathbb{R}$.

In general, any $*$-representation of $L^1(G)$ can be extended uniquely to a $*$-representation of $C^*(G)$, hence by Theorems 2.3 and 2.4, there is a one-to-one correspondence between unitary representations of G and nondegenerate $*$-representations of $C^*(G)$. Hence, the unitary dual space $\widehat{C^*(G)}$ of the $C^*(G)$ can be identified with the unitary dual $\widehat{G}$ of G.

On the other hand, if $\mathcal{A}$ is a group C*-algebra and $\widehat{\mathcal{A}}$ is its dual space, then one can use the Fourier transform $\mathcal{F}$ defined by

$$\mathcal{F}(a) = \hat{a} = (\pi(a))_{\pi \in \widehat{\mathcal{A}}}, \quad \text{for all } a \in \mathcal{A}$$

to analyze the C*-algebra in the following sense. Consider the algebra $\ell^\infty(\widehat{\mathcal{A}})$ of all bounded operator fields defined over $\widehat{\mathcal{A}}$ by

$$\ell^\infty(\widehat{\mathcal{A}}) := \{A = (A(\pi) \in \mathcal{B}(\mathcal{H}_\pi))_{\pi \in \widehat{\mathcal{A}}}; \|A\|_\infty = \sup_\pi \|A(\pi)\|_{\mathrm{op}} < \infty\}.$$

The space $\ell^\infty(\widehat{\mathcal{A}})$ itself is a C*-algebra and the Fourier transform is an injective, hence isometric, homomorphism from $\mathcal{A}$ into $\ell^\infty(\widehat{\mathcal{A}})$. The task is to recognize the elements of $\mathcal{F}(\mathcal{A})$, the image of $\mathcal{A}$ under the Fourier transform $\mathcal{F}$, inside the big C*-algebra $\ell^\infty(\widehat{\mathcal{A}})$. We know that the dual space has a natural topology called the *Fell topology* introduced in [3]. More precisely, a net $(\pi_k)_k \subset \widehat{\mathcal{A}}$ converges to $\pi \in \widehat{\mathcal{A}}$ if for every element ξ in the Hilbert space $\mathcal{H}_\pi$ of π, there is $\xi_k \in \mathcal{H}_{\pi_k}$ for every k such that

$$\lim_k \langle \pi_k(a)\xi_k, \xi_k \rangle = \langle \pi(a)\xi, \xi \rangle \quad \text{for all} \quad a \in \mathcal{A}.$$

Therefore, if an operator field $A = (A(\pi))_{\pi \in \widehat{\mathcal{A}}}$ is in $\mathcal{F}(\mathcal{A})$, then it must satisfy the relation

$$\lim_k \langle A(\pi_k)\xi_k, \xi_k \rangle = \langle A(\pi)\xi, \xi \rangle.$$

However, if $\mathcal{A}$ is not abelian, then the topology of $\widehat{\mathcal{A}}$ is in general not Hausdorff. A net in $\widehat{\mathcal{A}}$ can have many limit points and simultaneously many cluster points (see [3, 1, 5, 6] for details). On the other hand, for most C*-algebras, either its dual space is not known or if it is known, the topology of it is a mystery. Hence, in general it is difficult to describe the C*-algebra of a given group. But for some classes of locally compact groups G, the unitary dual $\widehat{G}$ and its topology has been determined, this is the case of exponential solvable Lie groups. A Lie group G is called *exponential* if the exponential mapping $\exp : \mathfrak{g} \to G$ from the Lie algebra $\mathfrak{g}$ to its Lie group G is a diffeomorphism. For instance, connected simply connected nilpotent Lie groups belong to this class and so do many more solvable Lie groups. By finding the right algebraic conditions which the elements in $\mathcal{F}(\mathcal{C} * (\mathcal{G}))$ inside $\ell^\infty(\widehat{G})$ must satisfy, in [7], the C*-algebra of $ax + b$-like groups has been described as an algebra of operator fields defined over the dual space. In [9], the C*-algebra of the Heisenberg group and of the threadlike groups have been described as well. Once there is a concrete description of the C*-algebra of a locally compact group, isomorphisms between two such C*-algebras can be studied (e.g. [8]).

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