ON A NUMERICAL SOLUTION OF THE LAPLACE EQUATION

Jasmina Veta Buralieva
(joint work with E. Hadzieva and K. Hadzi-Velkova Saneva)

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- **Classical Galerkin method for ODE**
- **Wavelets and MRA**
- **Wavelet-Galerkin method for ODE**
- **Transformation of the Laplace DE**
- **Application of W-G method on the Laplace DE**

Classical Galerkin metod for ODE

- *Sturm-Liouville equation*

$$Lu(t) \equiv -\frac{d}{dt}\left(g(t)\frac{du}{dt}\right) + h(t)u(t) = f(t),\ a \leq t \leq b \qquad (1)$$

with BC

$$u(a) = c, u(b) = d. \qquad (2)$$

- 1) $\{v_j\}$ - complete orthonormal system for $L^2([a,b])$
  2) every $v_j \in C^2([a,b])$
  3) $v_j(a) = c, v_j(b) = d$.
- Approximation u_s of the exact solution u

$$u_s = \sum_{k \in \Lambda} x_k v_k. \qquad (3)$$

- Criterion for coefficients x_k

$$< Lu_s, v_j >=< f, v_j >, \forall j \in \Lambda. \tag{4}$$

  - If we substitute the equation (3) in (4) we obtain

$$\sum_{k \in \Lambda} \langle Lv_k, v_j \rangle x_k = \langle f, v_j \rangle, \forall j \in \Lambda. \tag{5}$$

  - $A = [a_{j,k}]_{j,k \in \Lambda}$, $a_{j,k} = \langle Lv_k, v_j \rangle$;
    $X = (x_k)_{k \in \Lambda}$; $Y = (y_k)_{k \in \Lambda}$, $y_k = \langle f, v_k \rangle$

$$AX = Y. \tag{6}$$

  - Wavelet-Galerkin method: functions v_j are wavelets

Wavelets

- wavelet ψ: L^2 function which satisfy the admissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty. \tag{7}$$

The condition (7) implies that

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t) dt = 0.$$

- wavelets

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right), a > 0, b \in R.$$

Multiresolution analysis (MRA)

Multiresolution analysis of the space $L^2(R)$ consists of a sequence of closed subspace $\{V_j\}_{j=-\infty}^{\infty}$ with the following properties:

1. $V_j \subset V_{j+1}$
2. $\overline{\cup_{j \in Z} V_j} = L^2(R)$
3. $\cap_{j \in Z} V_j = \{0\}$
4. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$
5. $f(t) \in V_j \Leftrightarrow f(t-k) \in V_j, \forall k \in Z$
6. there exists a function ϕ (called *scaling function* or *father wavelet*) such that $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$, $k \in Z$ constitute orthonormal basis for corresponding subspace V_j.

- Let $\phi \in L^2(R)$ be compactly supported scaling function of MRA. Then
  1) $\int_{-\infty}^{\infty} \phi(t)dt \neq 0$
  2)$\phi(t) = \sum_{k \in Z} a_k \phi(2t-k)$, where a_k are real coefficients and $a_k \neq 0$ for only finitely many $k \in Z$ (the number of nonzero coefficients a_k is denoted by L).
  3) $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$, $j, k \in Z$ are orthonormal in $L^2(R)$ i.e.

$$\int_{-\infty}^{\infty} \phi(t-n)\phi(t-k)dt = \delta_{k,n}, \tag{8}$$

where

$$\delta_{n,k} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases}. \tag{9}$$

- One can construct wavelet ψ such that

$$\psi_{j,k}(t) = 2^{j/2}\phi(2^j t - k),\ j,k \in Z$$

constitute an orthonormal basis for $L^2(R)$.

- Daubechies scaling function

$$\phi(t) = \sum_{k=0}^{L-1} a_k \phi(2t - k) \tag{10}$$

- Daubechies wavelet function

$$\psi(t) = \sum_{k=2-L}^{1} (-1)^k a_{1-k} \phi(2t - k) \tag{11}$$

where L *is a positive even integer and denotes the genus of the Daubechies wavelet.*

Wavelet-Galerkin method for ODE

$$g(t)u''(t) + g'(t)u'(t) + h(t)u(t) = 0, \quad t \in [a, b], \tag{12}$$

with BC

$$u(a) = c, u(b) = d. \tag{13}$$

- Approximate solution

$$u_j(t) = \sum_{k=1-L}^{2^j} c_k \phi_{j,k}(t), k \in Z, \tag{14}$$

where ϕ is the scaling function of MRA.

Remark

- There are no closed-form formulas for the Daubechies wavelets and scaling functions.
- W-G method with Daubechies scaling functions: homogeneous differential equations.

- For $j = 0$ and $L = 4$

$$u_0(t) = \sum_{k=-3}^{1} c_k \phi(t-k), t \in [a, b]. \tag{15}$$

$$g(t)\frac{d^2}{dt^2}\sum_{k=-3}^{1} c_k \phi(t-k) + g'(t)\frac{d}{dt}\sum_{k=-3}^{1} c_k \phi(t-k) + $$
$$+h(t)\sum_{k=-3}^{1} c_k \phi(t-k) = 0. \tag{16}$$

Taking inner product with $\phi(t-n)$, $n \in \{-3,-2,-1,0,1\}$, we obtain

$$\sum_{k=-3}^{1} c_k \Omega_{n-k} + \sum_{k=-3}^{1} c_k a_{n,k} + \sum_{k=-3}^{1} c_k s_{n,k} = 0, \tag{17}$$

where

$$\Omega_{n-k} = \int_{-3}^{4} g(t)\phi^{''}(t-k)\phi(t-n)dt, \tag{18}$$

$$a_{n,k} = \int_{-3}^{4} g'(t)\phi^{'}(t-k)\phi(t-n)dt, \tag{19}$$

$$s_{n,k} = \int_{-3}^{4} h(t)\phi(t-k)\phi(t-n)dt. \tag{20}$$

- By using BC (13) we obtain

$$u_0(a) = \sum_{k=-3}^{1} c_k \phi(a-k) = c \tag{21}$$

and

$$u_0(b) = \sum_{k=-3}^{1} c_k \phi(b-k) = d \tag{22}$$

- We replace the first and the last equation of system (17) by (21) and (22), respectively and obtain the matrix equation

$$TC = B \tag{23}$$

$$T = \begin{bmatrix} \phi(a+3) & \phi(a+2) & \phi(a+1) & \phi(a) & \phi(a-1) \\ \Omega_{-2+3} + a_{-2,-3} + s_{-2,-3} & \Omega_{-2+2} + a_{-2,-2} + s_{-2,-2} & \Omega_{-2+1} + a_{-2,-1} + s_{-2,-1} & \Omega_{-2-0} + a_{-2,0} + s_{-2,0} & \Omega_{-2-1} + a_{-2,1} + s_{-2,1} \\ \Omega_{-1+3} + a_{-1,-3} + s_{-1,-3} & \Omega_{-1+2} + a_{-1,-2} + s_{-1,-2} & \Omega_{-1+1} + a_{-1,-1} + s_{-1,-1} & \Omega_{-1-0} + a_{-1,0} + s_{-1,0} & \Omega_{-1-1} + a_{-1,1} + s_{-1,1} \\ \Omega_{0+3} + a_{0,-3} + s_{0,-3} & \Omega_{0+2} + a_{0,-2} + s_{0,-2} & \Omega_{0+1} + a_{0,-1} + s_{0,-1} & \Omega_{0} + a_{0,0} + s_{0,0} & \Omega_{-1+0} + a_{0,1} + a_{0,1} \\ \phi(b+3) & \phi(b+2) & \phi(b+1) & \phi(b) & \phi(b-1) \end{bmatrix}$$

$$C = \begin{bmatrix} c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix}, \; B = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ b \end{bmatrix}.$$

Transformation of the Laplace DE

- The famous Laplace equation

$$\triangle u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{24}$$

with substitutions

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta,$$

has the form

$$\triangle u \equiv \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0. \tag{25}$$

Fourier method subsumes

$$u = u(r, \theta, \varphi)$$

can be represented

$$u(r, \varphi, \theta) = R(r)\Phi(\varphi)\Theta(\theta)$$

$$\Phi\Theta\frac{d}{dr}(r^2R') + R\Phi\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Theta') + \frac{1}{\sin^2\theta}R\Theta\Phi = 0,$$

$$\frac{1}{R}\cdot\frac{d}{dr}(r^2R') = -\left[\frac{1}{\Theta}\cdot\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}(\sin\theta\cdot\Theta') + \frac{1}{\sin^2\theta}\cdot\frac{\Phi''}{\Phi}\right], \qquad (26)$$

$$\frac{1}{R}\cdot\frac{d}{dr}(r^2R') = \lambda, \qquad \frac{1}{\Theta}\cdot\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}(\sin\theta\cdot\Theta') + \frac{1}{\sin^2\theta}\cdot\frac{\Phi''}{\Phi} = -\lambda. \quad (27)$$

- System of three ODEs

$$\begin{cases} \frac{1}{R}\cdot\frac{d}{dr}(r^2R') = \lambda \\ -\frac{\Phi''}{\Phi} = \mu \\ \frac{1}{\Theta}\cdot\sin\theta\cdot\frac{d}{d\theta}(\sin\theta\cdot\frac{d\Theta}{d\theta}) + \lambda\sin^2\theta = \mu \end{cases} \tag{28}$$

  - The first ODE is Cauchy-Euler equation

$$r^2R'' + 2rR' - \lambda R = 0. \tag{29}$$

Its exact solution is

$$R(r) = C_1r^n + C_2\frac{1}{r^{n+1}}$$

for

$$\lambda = n(n+1).$$

- The second ODE is Sturm-Liouville equation

$$\Phi'' + \mu\Phi = 0, \tag{30}$$

general solution

$$\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi,$$

for $\mu = m^2$, where $m = 1, 2, \ldots$.

- The third ODE is Sturm-Liouville equation

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \Theta\left(\lambda \sin^2\theta - \mu\right) = 0. \tag{31}$$

Its exact solution is

$$P_{n,m} = (1 - x^2)^{\frac{m}{2}} \frac{d^m P_{n(x)}}{dx^m} = \frac{(1 - x^2)^{\frac{m}{2}}}{n!2^n} \frac{d^{n+m}}{dx^{n+m}}[(x^2 - 1)^n], \tag{32}$$

for $\lambda = n(n+1)$ and $\mu = m^2$ where

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}[(x^2 - 1)^n], x = \cos\theta.$$

Application of the W-G method on the Laplace equation

$$\phi(t) = \begin{cases} \frac{1}{2}t^2, & t \in [0,1] \\ -t^2 - 3t - \frac{3}{2}, & t \in [1,2] \\ \frac{1}{2}t^2 - 3t + \frac{9}{2}, & t \in [2,3] \\ 0, & t \notin [0,3] \end{cases} \tag{33}$$

satisfies

$$\phi(t) = \frac{1}{4}\phi(2t) + \frac{3}{4}\phi(2t-1) + \frac{3}{4}\phi(2t-2) + \frac{1}{4}\phi(2t-3),$$

so $L = 4$.

1. $\lambda = 0$ and $\mu = 4$
   - The first ODE is

$$r^2 R'' + 2rR' = 0 \tag{34}$$

   with the BC: $R(1) = 1, R(3) = 0.$
     - Approximate solution is

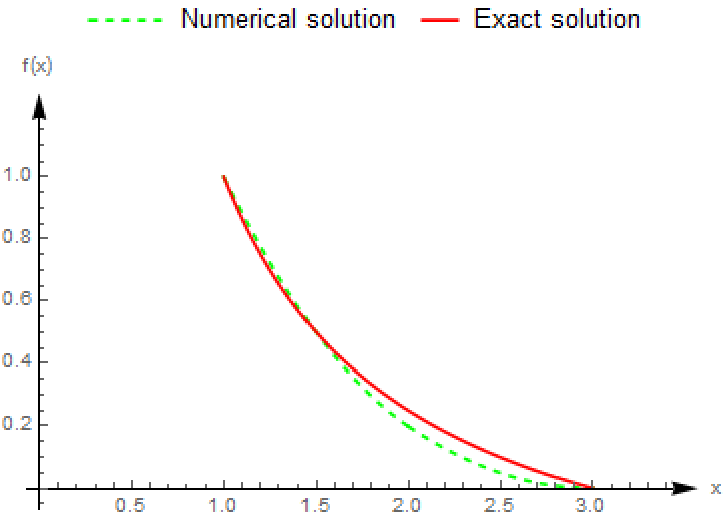
$$R_0(r) = \begin{cases} c_{-1}\phi(r+1) + c_0\phi(r) + c_1\phi(r-1), & r \in [1,2] \\ c_0\phi(r) + c_1\phi(r-1), & r \in [2,3] \end{cases}$$

     where

$$c_{-1} = \frac{690}{431}, c_0 = \frac{172}{431},$$
$$c_1 = 0.$$

Table: Comparison of results

Case t	numerical solution R_0	exact solution R	absolute error
1	1	1	0
1.1	0.883828	0.863636	0.0201919
1.2	0.775684	0.75	0.0256845
1.3	0.675568	0.653846	0.0217223
1.4	0.58348	0.571429	0.0120517
1.5	0.49942	0.5	0.000580046
1.6	0.423387	0.4375	0.0141125
1.7	0.355383	0.382353	0.0269701
1.8	0.295406	0.333333	0.0379273
1.9	0.243457	0.289474	0.0460166
2	0.199536	0.25	0.050464
2.1	0.161624	0.214286	0.0526616
2.2	0.127703	0.181818	0.0541152
2.3	0.0977726	0.152174	0.0544013
2.4	0.0718329	0.125	0.0531671
2.5	0.049884	0.1	0.050116
2.6	0.0319258	0.0769231	0.0449973
2.7	0.0179582	0.0555556	0.0375973
2.8	0.00798144	0.0357143	0.0277328
2.9	0.00199536	0.0172414	0.015246
3	0	0	0


Numerical solution
Exact solution
f(x)
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0
2.5
3.0
x

- The second ODE is

$$\Phi'' + 4\Phi = 0 \tag{35}$$

with BC: $\Phi(0) = 1, \Phi(\frac{\pi}{4}) = -1$.

  - Approximate solution is

$$\Phi_0(\varphi) = c_{-2}\phi(\varphi + 2) + c_{-1}\phi(\varphi + 1) + c_0\phi(\varphi), \quad \varphi \in [0, \frac{\pi}{4}]$$

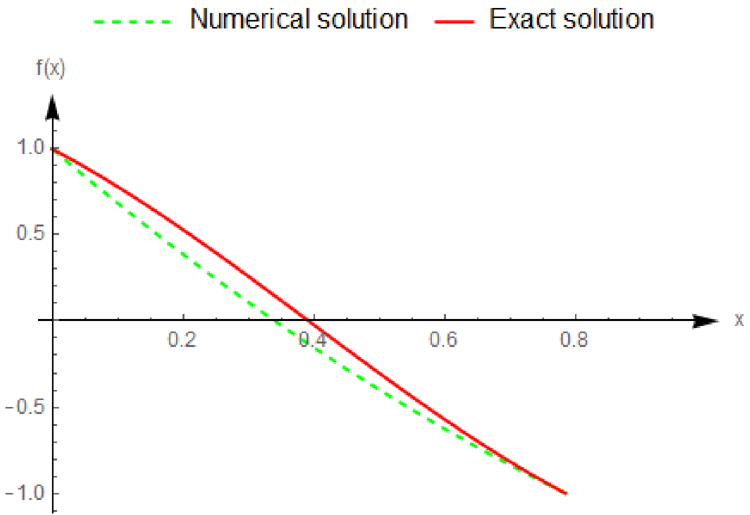
  where

$$c_{-2} = 2.61431, c_{-1} = -0.614306,$$

$$c_0 = -2.10591.$$

Table: Comparison of results

Case t	numerical solution Φ_0	exact solution Φ	absolute error
0	1	1	0
0.1	0.685824	0.781397	0.0955734
0.2	0.389018	0.531643	0.142625
0.3	0.109582	0.260693	0.151111
0.4	-0.152484	-0.0206494	0.131835
0.5	-0.39718	-0.301169	0.0960118
0.6	-0.624506	-0.569681	0.0548251
0.7	-0.834462	-0.815483	0.0189799
$\pi/4$	-1	-1	0


Numerical solution
Exact solution
f(x)
1.0
0.5
-0.5
-1.0
0.2
0.4
0.6
0.8
x

- The third ODE is

$$\sin^2(\theta)\Theta'' + \cos(\theta)\sin(\theta)\Theta' - 4\Theta = 0 \tag{36}$$

with BC: $\Theta(1) = 1, \Theta(2) = 2$.

2. For $\lambda = 0$ and $\mu = 2$

- The second ODE is

$$\Phi'' + 2\Phi = 0 \tag{37}$$

with BC: $\Phi(0) = 1, \Phi(\frac{\pi}{4}) = -1$.

  - Approximate solution is

$$\Phi_0(\varphi) = c_{-2}\phi(\varphi + 2) + c_{-1}\phi(\varphi + 1) + c_0\phi(\varphi), \quad \varphi \in [0, \frac{\pi}{4}]$$

  where

$$c_{-2} = 2.40053, c_{-1} = -0.400529, c_0 = -2.55335.$$

- The third ODE is

$$\sin^2(\theta)\Theta'' + \cos(\theta)\sin(\theta)\Theta' - 2\Theta = 0 \tag{38}$$

with BC: $\Theta(1) = 1, \Theta(2) = 2$.

  - Approximate solution is

$$\Theta_0(\theta) = c_{-1}\phi(\theta + 1) + c_0\phi(\theta) + c_1\phi(\theta - 1), \quad \theta \in [1, 2]$$

  where

$$c_{-1} = 2.427, c_0 = -0.426997,$$
$$c_1 = 4.427.$$

Table: Numerical results

Case t	numerical solution Φ_0	Case t	numerical solution Θ_0
0	1	1	1
0.1	0.723135	1.1	0.753141
0.2	0.452753	1.2	0.583361
0.3	0.188853	1.3	0.490661
0.4	-0.0685639	1.4	0.475041
0.5	-0.319499	1.5	0.536501
0.6	-0.563952	1.6	0.675041
0.7	-0.801922	1.7	0.890661
$\pi/4$	-1	1.8	1.18336
//	//	1.9	1.55314
//	//	2	2

3. $\lambda = 1$ and $\mu = 4$

- The first ODE is

$$r^2 R'' + 2rR' - R = 0 \tag{39}$$

with BC: $R(1) = 1, R(3) = 0$.

- Approximate solution is

$$R_0(r) = \begin{cases} c_{-1}\phi(r+1) + c_0\phi(r) + c_1\phi(r-1), & r \in [1,2] \\ c_0\phi(r) + c_1\phi(r-1), & r \in [2,3] \end{cases}$$

where

$$c_{-1} = -\frac{938}{499499}, c_0 = \frac{60}{499},$$
$$c_1 = 0.$$

- The third ODE is

$$\sin^2(\theta)\Theta'' + \cos(\theta)\sin(\theta)\Theta' + \Theta\left(\sin^2(\theta) - 4\right) = 0 \qquad (40)$$

with BC: $\Theta(1) = 1, \Theta(2) = 2$.

  - Approximate solution is

$$\Theta_0(\theta) = c_{-1}\phi(\theta+1) + c_0\phi(\theta) + c_1\phi(\theta-1), \quad \theta \in [1,2]$$

  where

$$c_{-1} = 3.27863, c_0 = -1.27863,$$
$$c_1 = 5.27863.$$

Table: Numerical results

Case t	numerical solution Φ_0	Case t	numerical solution Θ_0
1	1	1	1
1.2	0.680882	1.1	0.599847
1.4	0.427335	1.2	0.310839
1.6	0.239359	1.3	0.132976
1.8	0.116954	1.4	0.0662585
2	0.0601202	1.5	0.110686
2.2	0.038477	1.6	0.266258
2.4	0.0216433	1.7	0.532976
2.6	0.00961924	1.8	0.910839
2.8	0.00240481	1.9	1.39985
3	0	2	2

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