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On a Class of Differential Equations of Second Order with Polynomial Coefficients

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In this article we observe a class of linear differential equation of second order,

$$(x-x_1)(x-x_2)(x-x_3)y" + (b_2x^2+b_1x+b_0)y' + (c_1x+c_0)y = 0,$$

where $x_1, x_2, x_3, b_2, b_1, b_0, c_1, c_0$ are real numbers. Some substitutions for the given equation are introduced using the coefficient of y'' and with their help, some existence conditions for the integrability of the new equations are obtained, which contain a natural number.

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We observe the second order differential equation of the form

$$(1)\qquad (x-x_1)(x-x_2)(x-x_3)y" + (b_2x^2+b_1x+b_0)y' + (c_1x+c_0)y = 0,$$

where $x_1 \neq x_2 \neq x_3$.

In [1], it was already proven that, in order to have one polynomial solution, the equation (1) has to satisfy the conditions:

$$(2)\qquad \begin{aligned} &n^2+(b_2-1)n+c_1=0,\\ &b_0+x_2x_3-(x_2+x_3)x_1+(b_2+1)x_1^2+b_1x_1=0,\\ &c_0b_1+c_0^2-c_1b_0+(c_1+b_2)(c_1x_1+2c_0)x_1=0. \end{aligned}$$

(The first equation in (2) is the characteristic equation of the differential equation (1).) In such a case, the polynomial solution of (1) is given with the formula

$$y = e^{-F}\left[(x+K)(x-x_2)^{n-1}(x-x_3)^{n-1}e^{F}\right]^{(n-1)}, \tag{3}$$

where

$$\begin{aligned} &F = \int \frac{Mx+N}{(x-x_1)(x-x_2)}dx, \quad M = b_2 - 1,\\ &N = b_1 + x_1 + x_1 b_2, \quad K = -\frac{x_1c_1 + n(b_1 + 2x_1b_2 + x_1c_1 + c_0)}{c_1}. \end{aligned} \tag{4}$$

Now, we will obtain the general solution of (1), using different procedure than the one used before. To this purpose, we start with the equation

$$\begin{aligned} &(Sx+T)(x+K)(x^2+Qx+R)u'' + [\beta_2x^2 + \beta_1x + \beta_0) + 2(Sx+T)\cdot\\ &\cdot(x^2+Qx+R) + (n-1)(x+K)(Sx+T)(2x+q) -\\ &-(n-1)S(x+K)(x_2+Qx+R)]u' = 0 \end{aligned}$$

Using the substitution $v = (x+K)(x^2+Qx+R)^{n-1}e^{\int \frac{Mx+N}{x^2+Qx+R}dx}u$ and according to conditions (2′), we get the transformed equation

$$\begin{aligned} &(Sx+T)(x^2+Qx+R)v'' + \{-2(Sx+T)(Mx+N) + (\beta_2x_2 + \beta_1x + \beta_0)\\ &-(n-1)\left[S(x^2+Qx+R) + (Sx+T)(2x+Q)\right]\}\, v' + \{(Mx+N)\\ &\times(2S + MS - \beta_2) + \beta_1x + \beta_0 - M(Sx+T) - (n-1)\,[-2S(Mx+N)\\ &-2M(Sx+T) + 2\beta_2x + \beta_1] + n(n-1)\,[S(2x+Q) + (Sx+T)]\}\, v = 0. \end{aligned}$$

After differentiating the last equation *n-1* times and putting $v^{(n-1)} = z$, we obtain the equation

$$\begin{aligned} &(Sx+T)(x^2+Qx+R)z'' + [-2(Sx+T)(Mx+N) + (\beta_2x_2 + \beta_1x + \beta_0)]\, z'\\ &+ [(Mx+N)(2S + MS - \beta_2) + \beta_1x + \beta_0 - M(Sx+T)]\, z = 0. \end{aligned}$$

Finally, the last substitution $z = e^{\int \frac{Mx+N}{x^2+Qx+R}dx}y$ leads to an equation of the same type as (1),

$$(Sx+T)(x^2+Qx+R)y'' + (\beta_2x_2 + \beta_1x + \beta_0)y' + (\beta_1x + \beta_0)y = 0. \tag{1′}$$

Now according to the above transformations, we can state the general solution of (1′):

$$(4') \qquad \begin{aligned} y &= e^{-F}\Big\{(x+K)(x^2+Qx+R)^{n-1}e^F \\ &\times \Big[C_1 + C_2\int (Sx+T)^{n-1}(x^2+Qx+R)^{1-n}(x+K)^{-2}dx\Big]\Big\}^{(n-1)}, \end{aligned}$$

where

$$F = \int \frac{Mx+N}{x^2+Qx+R}dx, \quad M = \frac{\beta_2 - S}{S},$$
$$N = \frac{\beta_1 - T}{S} - \frac{T\beta_2}{S^2}, \quad K = \frac{T\gamma_1 - n(S\beta_1 - 2T\beta_2 - T\gamma_1 + S\gamma_0)}{S\gamma_1}.$$

The conditions for the existence of one polynomial solution of (1') are given with:

$$(2') \qquad \begin{aligned} &Sn^2 + (\beta^2 - S)n + \gamma_1 = 0 \\ &S^2(\beta_0 - TQ + SR) + T^2(\beta_2 + S) - T\beta_1 S = 0 \\ &S^2(\gamma_0\beta_1 + \gamma_0^2 - \gamma_1\beta_0) + T(\gamma_1 + \beta_2)(T\gamma_1 - 2S\gamma_0) = 0. \end{aligned}$$

If we put $S = 1$, $T = -x_1$, $Q = -(x_2 + x_3)$, $R = x_2x_3$, where $x^2 + Qx + R = (x - x_1)(x - x_3)$, we obtain a general solution formula for the equation (1),

$$(4'') \qquad \begin{aligned} y &= e^{-F}\Big\{(x+K)(x-x_2)^{n-1}(x-x_3)^{n-1}e^F \\ &\times \Big[C_1 + C_2\int (x-x_1)^{n-1}(x-x_2)^{1-n}(x-x_3)^{1-n}(x+K)^{-2}dx\Big]\Big\}^{(n-1)}. \end{aligned}$$

Theorem 1. *If the conditions* (2) *to the differential equation* (1) *are satisfied, then this equation has one polynomial solution given with the formula* (3)*, while the general solution is given with the formula* (4'')*.*

Now, we introduce substitutions to the equation (1) which are given with

$$(5) \qquad y = (x - x_1)^\alpha (x - x_2)^\beta (x - x_3)^\gamma z.$$

It was already proven in [2] that, applying these substitutions, we can obtain at most seven other differential equations of the same type as (1):

$$(6) \qquad (x - x_1)(x - x_2)(x - x_3)z_1'' + (b_2^1x^2 + b_1^1x + b_0^1)z_1' + (c_1^1x + c_0^1)z_1 = 0$$

$$(7) \qquad (x - x_1)(x - x_2)(x - x_3)z_2'' + (b_2^2x^2 + b_1^2x + b_0^2)z_2' + (c_1^2x + c_0^2)z_2 = 0$$

$$(8)\qquad (x-x_1)(x-x_2)(x-x_3)z_3''+(b_2^3x^2+b_1^3x+b_0^3)z_3'+(c_1^3x+c_0^3)z_3=0$$

$$(9)\qquad (x-x_1)(x-x_2)(x-x_3)z_4''+(b_2^4x^2+b_1^4x+b_0^4)z_4'+(c_1^4x+c_0^4)z_4=0$$

$$(10)\qquad (x-x_1)(x-x_2)(x-x_3)z_5''+(b_2^5x^2+b_1^5x+b_0^5)z_5'+(c_1^5x+c_0^5)z_5=0$$

$$(11)\qquad (x-x_1)(x-x_2)(x-x_3)z_6''+(b_2^6x^2+b_1^6x+b_0^6)z_6'+(c_1^6x+c_0^6)z_6=0$$

$$(12)\qquad (x-x_1)(x-x_2)(x-x_3)z_7''+(b_2^7x^2+b_1^7x+b_0^7)z_7'+(c_1^7x+c_0^7)z_7=0$$

Here we have used the following groups of indexed coefficients:

$$\begin{aligned}
&b_2^1=2\alpha+b_2,\quad b_1^1=-(2\alpha x_3+2\alpha x_2-b_1),\quad b_0^1=2\alpha x_2x_3+b_0,\\
&c_1^1=\alpha(\alpha-1)+\alpha b_2+c_1,\\
&c_0^1=\alpha(\alpha-1)(x_1-x_2-x_3)+\alpha(b_2x_1+b_1)+c_0,
\end{aligned}$$

$$\begin{aligned}
&b_2^2=2\beta+b_2,\quad b_1^2=-(2\beta x_1+2\beta x_3-b_1),\quad b_0^2=2\beta x_1x_3+b_0,\\
&c_1^2=\beta(\beta-1)+\beta b_2+c_1,\\
&c_0^2=\beta(\beta-1)(x_2-x_3-x_1)+\beta(b_2x_2+b_1)+c_0,
\end{aligned}$$

$$\begin{aligned}
&b_2^3=2\gamma+b_2,\quad b_1^3=-(2\gamma x_1+2\gamma x_2-b_1),\quad b_0^3=2\gamma x_1x_2+b_0,\\
&c_1^3=\gamma(\gamma-1)+\gamma b_2+c_1,\\
&c_0^3=\gamma(\gamma-1)(x_3-x_2-x_1)+\gamma(b_2x_3+b_1)+c_0,
\end{aligned}$$

$$\begin{aligned}
&b_2^4=2\alpha+2\beta+b_2,\quad b_1^4=-(2\alpha x_3+2\alpha x_2+2\beta x_1+2\beta x_3-b_1),\\
&b_0^4=2\alpha x_2x_3+2\beta x_1x_3+b_0,\\
&c_1^4=2\alpha\beta+\alpha(\alpha-1)+\beta(\beta-1)+(\alpha+\beta)b_2+c_1,\\
&c_0^4=-2\alpha\beta x_3+\alpha(\alpha-1)(x_1-x_2-x_3)+\beta(\beta-1)(x_2-x_3-x_1)\\
&+\alpha(b_2x_1+b_1)+\beta(b_2x_2+b_1)+c_0,
\end{aligned}$$

$$\begin{aligned}
&b_2^5=2\alpha+2\gamma+b_2,\quad b_1^5=-(2\alpha x_3+2\alpha x_2+2\gamma x_1+2\gamma x_2-b_1),\\
&b_0^5=2\alpha x_2x_3+2\gamma x_1x_2+b_0,\\
&c_1^5=2\alpha\gamma+\alpha(\alpha-1)+\gamma(\gamma-1)+(\alpha+\gamma)b_2+c_1,\\
&c_0^5=-2\alpha\gamma x_2+\alpha(\alpha-1)(x_1-x_2-x_3)+\gamma(\gamma-1)(x_3-x_2-x_1)
\end{aligned}$$

$$+\alpha(b_2x_1+b_1)+\gamma(b_2x_3+b_1)+c_0,$$

$$\begin{aligned}
&b_2^6=2\beta+2\gamma+b_2,\quad b_1^6=-(2\beta x_1+2\beta x_3+2\gamma x_1+2\gamma x_2-b_1),\\
&b_0^6=2\beta x_1x_3+2\gamma x_1x_2+b_0,\\
&c_1^6=2\beta\gamma+\beta(\beta-1)+\gamma(\gamma-1)+(\beta+\gamma)b_2+c_1,\\
&c_0^6=-2\beta\gamma x_1+\beta(\beta-1)(x_2-x_3-x_1)+\gamma(\gamma-1)(x_3-x_2-x_1)\\
&+\beta(b_2x_2+b_1)+\gamma(b_2x_3+b_1)+c_0,
\end{aligned}$$

$$\begin{aligned}
&b_2^7=2\alpha+2\beta+2\gamma+b_2,\\
&b_1^7=-(2\alpha x_3+2\alpha x_2+2\beta x_1+2\beta x_3+2\gamma x_1+2\gamma x_2-b_1),\\
&b_0^7=2\alpha x_2x_3+2\beta x_1x_3+2\gamma x_1x_2+b_0,\\
&c_1^7=2\alpha\beta+2\beta\gamma+2\alpha\gamma+\alpha(\alpha-1)+\beta(\beta-1)+\gamma(\gamma-1)+(\alpha+\beta+\gamma)b_2+c_1,\\
&c_0^7=-2\alpha\beta x_3-2\alpha\gamma x_2-2\beta\gamma x_1+\alpha(\alpha-1)(x_1-x_2-x_3)\\
&+\beta(\beta-1)(x_2-x_3-x_1)+\gamma(\gamma-1)(x_3-x_2-x_1)\\
&+\alpha(b_2x_1+b_1)+\beta(b_2x_2+b_1)+\gamma(b_2x_3+b_1)+c_0,
\end{aligned}$$

where

$$\begin{aligned}
&\alpha=1-\frac{b_2x_1^2+b_1x_1+b_0}{(x_3-x_1)(x_2-x_1)},\qquad \beta=1-\frac{b_2x_2^2+b_1x_2+b_0}{(x_1-x_2)(x_3-x_2)},\\
&\gamma=1-\frac{b_2x_3^2+b_1x_3+b_0}{(x_1-x_3)(x_2-x_3)},
\end{aligned}\tag{13}$$

and the substitutions for each of the equations are adequately given with

$$\begin{aligned}
&y=(x-x_1)^\alpha z_1,\quad y=(x-x_2)^\beta z_2,\quad y=(x-x_3)^\gamma z_3\\
&y=(x-x_1)^\alpha(x-x_2)^\beta z_4,\quad y=(x-x_1)^\alpha(x-x_3)^\gamma z_5,\\
&y=(x-x_2)^\beta(x-x_3)^\gamma z_6,\quad y=(x-x_1)^\alpha(x-x_2)^\beta(x-x_3)^\gamma z_7.
\end{aligned}\tag{13'}$$

The conditions for the existence of one polynomial solution for each of the above equations (6)-(12), according to the previously stated, are:

$$\begin{aligned}
&n^2+(b_2^1-1)n+c_1^1=0,\\
&b_0^1+x_2x_3-(x_2+x_3)x_1+(b_2^1+1)x_1^2+b_1^1x_1=0,\\
&c_0^1b_1^1+(c_0^1)^2-c_1^1b_0^1+(c_1^1+b_2^1)(c_1^1x_1+2c_0^1)x_1=0,
\end{aligned}\tag{6'}$$

$$(7') \quad \begin{aligned} &n^2 + (b_2^2 - 1)n + c_1^2 = 0,\\ &b_0^2 + x_2 x_3 - (x_2 + x_3)x_1 + (b_2^2 + 1)x_1^2 + b_1^2 x_1 = 0,\\ &c_0^2 b_1^2 + (c_0^2)^2 - c_1^2 b_0^2 + (c_1^2 + b_2^2)(c_1^2 x_1 + 2c_0^2)x_1 = 0, \end{aligned}$$

$$(8') \quad \begin{aligned} &n^2 + (b_2^3 - 1)n + c_1^3 = 0,\\ &b_0^3 + x_2 x_3 - (x_2 + x_3)x_1 + (b_2^3 + 1)x_1^2 + b_1^3 x_1 = 0,\\ &c_0^3 b_1^3 + (c_0^3)^2 - c_1^3 b_0^3 + (c_1^3 + b_2^3)(c_1^3 x_1 + 2c_0^3)x_1 = 0, \end{aligned}$$

$$(9') \quad \begin{aligned} &n^2 + (b_2^4 - 1)n + c_1^4 = 0,\\ &b_0^4 + x_2 x_3 - (x_2 + x_3)x_1 + (b_2^4 + 1)x_1^2 + b_1^4 x_1 = 0,\\ &c_0^4 b_1^4 + (c_0^4)^2 - c_1^4 b_0^4 + (c_1^4 + b_2^4)(c_1^4 x_1 + 2c_0^4)x_1 = 0, \end{aligned}$$

$$(10') \quad \begin{aligned} &n^2 + (b_2^5 - 1)n + c_1^5 = 0,\\ &b_0^5 + x_2 x_3 - (x_2 + x_3)x_1 + (b_2^5 + 1)x_1^2 + b_1^5 x_1 = 0,\\ &c_0^5 b_1^5 + (c_0^5)^2 - c_1^5 b_0^5 + (c_1^5 + b_2^5)(c_1^5 x_1 + 2c_0^5)x_1 = 0, \end{aligned}$$

$$(11') \quad \begin{aligned} &n^2 + (b_2^6 - 1)n + c_1^6 = 0,\\ &b_0^6 + x_2 x_3 - (x_2 + x_3)x_1 + (b_2^6 + 1)x_1^2 + b_1^6 x_1 = 0,\\ &c_0^6 b_1^6 + (c_0^6)^2 - c_1^6 b_0^6 + (c_1^6 + b_2^6)(c_1^6 x_1 + 2c_0^6)x_1 = 0, \end{aligned}$$

$$(12') \quad \begin{aligned} &n^2 + (b_2^7 - 1)n + c_1^7 = 0,\\ &b_0^7 + x_2 x_3 - (x_2 + x_3)x_1 + (b_2^7 + 1)x_1^2 + b_1^7 x_1 = 0,\\ &c_0^7 b_1^7 + (c_0^7)^2 - c_1^7 b_0^7 + (c_1^7 + b_2^7)(c_1^7 x_1 + 2c_0^7)x_1 = 0, \end{aligned}$$

and the formula for each of the related polynomial solutions is given with

$$(14) \qquad z_i = e^{-F_i}\left[(x + K_i)(x - x_2)^{n-1}(x - x_3)^{n-1} e^{F_i}\right]^{(n-1)},$$

while the related general solution is given with

$$(15) \qquad z_i = e^{-F_i}\Big\{(x + K_i)(x - x_2)^{n-1}(x - x_3)^{n-1} e^{F_i}$$

$$\times \left[C_1 + C_2 \int (x - x_1)^{n-1}(x - x_2)^{1-n}(x - x_3)^{1-n}(x + K_i)^{-2} dx\right]\Big\}^{(n-1)},$$

$$i = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,$$

where

$$F_i = \int \frac{M_i x + N_i}{(x - x_2)(x - x_3)} dx, \quad M_i = b_2^i - 1, \quad N_i = b_1^i + x_1 + x_1 b_2^i,$$
$$K_i = -\frac{x_1 c_1^i + n(b_1^i + 2x_1 b_2^i + x_1 c_1^i + c_0^i)}{c_1}, \quad i = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7.$$

Theorem 2. *Given the differential equation* (1)*, and if there is a number $n \in \mathbb{N}$ (the smaller if there are two nulls of the characteristic equation) for which at least one of the conditions* (2), (6′), (7′), (8′), (9′), (10′), (11′) *and* (12′) *hold, then the equation* (1) *is solvable in closed form. In addition, with* (13′) *and* (14) *we obtain the formula for one of its particular solutions,while* (15) *gives the formula for the general solution, while* (15) *gives the formula for the general solution.*

Other existence conditions for integrability of (1) are obtained in [3, 4].

Example. We consider the differential equation

$$(x - 1)(x + 1)(x - 3)y'' - (3x^2 + 6x - 1)y' + 4(x + 1)y = 0.$$

It has one polynomial solution

$$y_1 = (x + 1)^2,$$

while the general solution is

$$y = C_1\left((x + 1)^2 ln(x + 1) + 2\right) + C_2(x + 1)^2. \tag{16}$$

Having $\alpha = \beta = \gamma = 2$ and using the appropriate substitutions, one can obtain seven different transformational equations:

$$(x - 1)(x + 1)(x - 3)z_1'' + (x^2 + 10x + 13)z_1' - 4z_1 = 0,$$
$$(x - 1)(x + 1)(x - 3)z_2'' + (x^2 + 2x - 11)z_2' = 0,$$
$$(x - 1)(x + 1)(x - 3)z_3'' + (x^2 - 6x - 3)z_2' + 4z_3 = 0,$$
$$(x - 1)(x + 1)(x - 3)z_4'' + (5x^2 + 18x + 1)z_4' + (4x + 20)z_4 = 0,$$
$$(x - 1)(x + 1)(x - 3)z_5'' + (5x^2 + 10x + 9)z_5' + (4x + 4)z_5 = 0,$$
$$(x - 1)(x + 1)(x - 3)z_6'' + (5x^2 + 2x - 15)z_6' + (4x - 8)z_6 = 0,$$
$$(x - 1)(x + 1)(x - 3)z_7'' + (9x^2 - 18x - 3)z_7' + (16x + 16)z_7 = 0.$$

Their general solutions can be obtained from (16), according to the related substitutions.

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