8 МСДР 2004, (99 - 108)
Зборник на трудови
30.09.- 03.10.2004 год.
Охрид, Македонија

ISBN 9989 – 630 – 49 – 6
COBISS.MK – ID 61901322

ON INTEGRABILITY OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract: In this article we observe a class of linear differential equation of third order, which is obtained from a class of second order differential equation. Using some previous results, conditions for the existence of two of its particular solutions are obtained.

In this article we observe a differential equation of type

$$(x-x_1)(x-x_2)(x-x_3)\, y''' + (B_2 x^2 + B_1 x + B_0)\, y'' + \\ + (C_1 x + C_0)\, y' + D\, y = 0 \qquad (1)$$

Using differentiation and other relevant results, we obtain the conditions for existence and integrability of (1), together with the formulae of its two particular solutions.

We start with differentiation of the differential equation of second order with polynomial coefficients

$$(x-x_1)(x-x_2)(x-x_3)\, y'' + (b_2 x^2 + b_1 x + b_0)\, y' + \\ + (c_1 x + c_0)\, y = 0 \qquad (2)$$

thus obtaining the differential equation of third order

$$(x-x_1)(x-x_2)(x-x_3)\, y''' + [(3+b_2)\, x^2 + (-2x_1-2x_2-2x_3+b_1)\, x + \\ + x_1x_2+x_1x_3+x_2x_3+b_0\,]\, y'' + [(2b_2 + c_1)\, x + b_1 + c_0]\, y' + c_1\, y = 0 \qquad (3)$$

which is of type (1). Using this fact, we get the interdepedence equations between the coefficients of (1) i (3):

$$D = c_1$$
$$C_0 = b_1 + c_0$$
$$C_1 = 2b_2 + c_1$$
$$B_0 = x_1x_2 + x_1x_3 + x_2x_3 + b_0$$
$$B_1 = -2(x_1 + x_2 + x_3) + b_1$$
$$B_2 = 3 + b_2$$

i.e.

$$\begin{aligned} b_2 &= B_2 - 3 \\ b_1 &= B_1 + 2(x_1 + x_2 + x_3) \\ b_0 &= B_0 - x_1x_2 - x_1x_3 - x_2x_3 \\ c_0 &= C_0 - B_1 - 2(x_1 + x_2 + x_3) \\ c_1 &= D \\ c_1 &= C_1 - 2B_2 + 6 \end{aligned} \qquad (*)$$

In the article [1] we obtain the conditions for existence of one particular polynomial solution of (1), of order n. Using this conditions together with (*) over (1), we get the conditions for existence of one particular polynomial solution of the differential equation (1):

$$\begin{gathered} n^2 + (B_2 - 4)\, n + D = 0 \\ B_2\, x_1^2 + B_1\, x_1 + B_0 = 0 \\ [C_0 - B_1 - 2\,(x_1 + x_2 + x_3)]\, C_0 + D\,(x_1x_2 + x_1x_3 + x_2x_3 - B_0) + \\ (D + B_2 - 3) + \{Dx_1 + 2[C_0 - B_1 - 2(x_1 + x_2 + x_3)]\}\, x_1 = 0 \\ 2B_2 - C_1 + D = 6 \end{gathered} \qquad (**)$$

where the last equation is an additional condition, a result of the last two equations of (*).

In this, the polynomial degree $n \in N$ of the particular solution, is a root of the characteristic equation given with the first relation of (**), the lower one if both roots are natural numbers.

The formula of the polynomial solution is given with

$$y_1 = e^{-F}\left[(x+K)(x-x_2)^{n-1}(x-x_3)^{n-1}e^F\right]^{(n-1)} \qquad (4)$$

where

$$F=\int\frac{Mx+N}{(x-x_1)(x-x_2)}dx,$$

$$M=B_2-4\,,\quad N=B_1+2(x_2+x_3)+x_1B_2\,,\quad K=-\frac{x_1D+n(C_1x_1+D)}{D}\,.$$

In $[2]$, the formula of the general solution of (2) is given, from which we can find the second particular solution of (1), according to the formula (5) **:**

$$y_2=e^{-F}\Big[(x+K)(x-x_2)^{n-1}(x-x_3)^{n-1}e^F\cdot$$
$$\cdot\int(x-x_1)^{n-1}(x-x_2)^{1-n}(x-x_3)^{1-n}(x+K)^{-2}dx\ \Big]^{(n-1)}\qquad(5)$$

where K is as stated above. We obtain the third particular solution using the classical method.

*Theorem*1: If the differential equation (1) satisfies the conditions (**), then it has two independent particular solutions given with the formulae (4) and (5), of which the first is a polynomial of order *n*, the lower one if the characteristic equation of (2) has two natural roots.

According to $[2]$, the differential equation (1) can be transformed to at most 7 other differential equations of the same type, using the substitution

$$y=(x-x_1)^{\alpha}(x-x_2)^{\beta}(x-x_3)^{\gamma}z,\quad\text{where}$$

$$\alpha=1-\frac{b_2x_1^{\ 2}+b_1x_1+b_0}{(x_3-x_1)(x_2-x_1)},\ \beta=1-\frac{b_2x_2^{\ 2}+b_1x_2+b_0}{(x_1-x_2)(x_3-x_2)},\ \gamma=1-\frac{b_2x_3^{\ 2}+b_1x_3+b_0}{(x_1-x_3)(x_2-x_3)}\qquad(6)$$

With the connecting relations between the equations of second and third order, together with the substitutions :

$$y=(x-x_1)^{\alpha}(x-x_2)^{\beta}(x-x_3)^{\gamma}z_7,\ y=(x-x_1)^{\alpha}z_1,\ y=(x-x_2)^{\beta}z_2\,,$$
$$y=(x-x_3)^{\gamma}z_3\qquad y=(x-x_1)^{\alpha}(x-x_2)^{\beta}z_4$$
$$y=(x-x_1)^{\alpha}(x-x_3)^{\gamma}z_5\qquad y=(x-x_2)^{\beta}(x-x_3)^{\gamma}z_6\qquad(7)$$

we obtain at most seven other differential equations of third order with one polynomial solution.

The same equations, with the use of the coefficients of (1), have the following form:

$$(x-x_1)(x-x_2)(x-x_3)\, \mathbf{z_1'''}+[(2\alpha+B_2)\, x^2 - (2\alpha x_3+2\alpha x_2-B_1)\, x +2\alpha x_2x_3+B_0]\, \mathbf{z_1''}+ \\ + \{[\alpha(\alpha-1)+4\alpha+\alpha(B_2-3)+C_1]\, x + \alpha(\alpha-1)(x_1-x_2-x_3) + \\ + \alpha[(B_2-3)x_1+B_1 + 2x_1+2x_2+2x_3]-2\alpha(x_2+x_3) +C_0\}\, \mathbf{z_1'}+ \\ + [\alpha(\alpha-1)+\alpha(B_2-3)+D]\, \mathbf{z_1} = 0 \qquad \mathbf{(1a)}$$

$$(x-x_1)(x-x_2)(x-x_3)\, \mathbf{z_2'''}+ [(2\beta+B_2)x^2 -(2\beta x_1+2\beta x_3 - B_1)\, x +2\beta x_3x_1+B_0]\, \mathbf{z_2''}+ \\ + \{[\beta(\beta-1)+ 4\beta+ \beta(B_2-3)+C_1]\, x + \beta(\beta-1)(x_2-x_3-x_1) + \beta[(B_2-3)x_2+ \\ + B_1+2x_1+2x_2+2x_3] – 2\beta(x_1+x_3) + C_0\}\, \mathbf{z_2'} + \\ + [\beta(\beta-1) + \beta(B_2-3) + D]\, \mathbf{z_2} = 0 \qquad \mathbf{(1b)}$$

$$(x-x_1)(x-x_2)(x-x_3)\mathbf{z_3'''}+[(2\gamma+B_2)x^2-(2\gamma x_2+2\gamma x_1-B_1)x+2\gamma x_1x_2+B_0]\mathbf{z_3''} + \\ + \{[\gamma(\gamma-1) + 4\gamma + \gamma(B_2-3) + C_1]x + \gamma(\gamma-1)(x_3-x_1-x_2) + \\ + \gamma[(B_2-3)x_3+ B_1+2x_1+2x_2+2x_3]–2\gamma(x_1+x_2)+C_0\}\, \mathbf{z_3'}+ \\ + [\gamma(\gamma-1)+\gamma(B_2-3)+D]\mathbf{z_3} = 0 \qquad \mathbf{(1c)}$$

$$(x-x_1)(x-x_2)(x-x_3)\, \mathbf{z_4'''}+ [(2\alpha+2\beta+B_2)\, x^2 – \\ - (2\alpha x_3+2\alpha x_2+2\beta x_1+2\beta x_3 -B_1)\, x + 2\alpha x_2x_3 + 2\beta x_3x_1 + B_0]\, \mathbf{z_4''} + \\ + \{[B_2(\alpha+\beta) + (\alpha+\beta)^2 + C_1]\, x + \alpha^2(x_1-x_2-x_3) + \beta^2(x_2-x_3-x_1) - 2\alpha\beta x_3 + \\ +(\alpha+\beta)(x_1+x_2+x_3+B_1) + (B_2-3)(\alpha x_1+\beta x_2) +C_0\}\, \mathbf{z_4'} + \\ + [(\alpha+\beta)^2 + (\alpha+\beta)(B_2-4) + D]\, \mathbf{z_4} = 0 \qquad \mathbf{(1d)}$$

$$(x-x_1)(x-x_2)(x-x_3)\, \mathbf{z_5'''} + [(2\alpha+2\gamma+B_2)\, x^2 – \\ - (2\alpha x_3+2\alpha x_2+2\gamma x_1+2\gamma x_2 -B_1)\, x + 2\alpha x_2x_3 + 2\gamma x_1x_2 + B_0]\, \mathbf{z_5''} + \\ + \{[B_2 (\alpha+\gamma) + (\alpha+\gamma)^2 + C_1]\, x + \alpha^2(x_1-x_2-x_3) + \gamma^2(x_3-x_2-x_1) - 2\alpha\gamma x_2 + \\ + (\alpha+\gamma)(x_1+x_2+x_3+B_1) + (B_2-3)(\alpha x_1+\gamma x_3) +C_0\}\, \mathbf{z_5'} + \\ + [(\alpha+\gamma)^2 + (\alpha+\gamma)(B_2-4) + D]\, \mathbf{z_5} = 0 \qquad \mathbf{(1e)}$$

$$(x-x_1)(x-x_2)(x-x_3)\, \mathbf{z_6'''} + [(2\beta+2\gamma+B_2)\, x^2 – \\ - (2\beta x_1+2\beta x_3+2\gamma x_1+2\gamma x_2 -B_1)\, x + 2\beta x_1x_3 + 2\gamma x_1x_2 + B_0]\, \mathbf{z_6''} + \\ + \{[B_2(\beta+\gamma) + (\beta+\gamma)^2 +C_1]\, x + \beta^2(x_2-x_1-x_3) + \gamma^2(x_3-x_2-x_1) - 2\beta\gamma x_1 + \\ + (\beta+\gamma)(x_1+x_2+x_3+B_1) + (B_2-3)(\beta x_2+\gamma x_3) +C_0\}\, \mathbf{z_6'} + \\ + [(\beta+\gamma)^2 + (\beta+\gamma)(B_2-4) + D]\, \mathbf{z_6} = 0 \qquad \mathbf{(1f)}$$

$$(x-x_1)(x-x_2)(x-x_3)\,\mathbf{z_7'''} + [(2\alpha+2\beta+2\gamma+B_2)\,x^2 -$$
$$-(2\alpha x_3+2\alpha x_2+2\beta x_1+2\beta x_3+2\gamma x_1+2\gamma x_2-B_1)x+2\alpha x_2x_3+2\beta x_3x_1+2\gamma x_2x_1+$$
$$+B_0]\,\mathbf{z_7''} + \{[\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\alpha\gamma+2\beta\gamma+(\alpha+\beta+\gamma)B_2+C_1]\,x -$$
$$-2\alpha x_3-2\alpha x_2-2\beta x_1-2\beta x_3-2\gamma x_1-2\gamma x_2-2\alpha\beta x_3-2\beta\gamma x_1-2\alpha\gamma x_2+$$
$$+\alpha(\alpha-1)(x_1-x_2-x_3)+\beta(\beta-1)(x_2-x_3-x_1)+\gamma(\gamma-1)(x_3-x_2-x_1)+$$
$$+\alpha[(B_2-3)x_1+b_1]+\beta[(B_2-3)x_2+b_1]+\gamma[(B_2-3)x_3+b_1]+C_0\}\,\mathbf{z_7'}+$$
$$+[2\alpha\beta+2\alpha\gamma+2\beta\gamma+\alpha^2+\beta^2+\gamma^2+(\alpha+\beta+\gamma)(B_2-4)+D]\,\mathbf{z_7}=0 \qquad \mathbf{(1g)}$$

The conditions for the existence of one particular polynomial solution for each of the equations, using the coefficients of (1), have a common fourth equation, same as the one in (**), are adequatly given with:

- $n^2+(2\alpha+B_2-4)\,n+\alpha(\alpha-1)+\alpha(B_2-3)+$
  $\alpha(\alpha-1)+\alpha(B_2-3)+D=0$ **(1a-u)**
- $B_2\,x_1^{\,2}+B_1\,x_1+2\alpha(x_1^{\,2}-x_1x_2+x_2x_3-x_1x_3)+B_0=0$
- $[\alpha(\alpha-1)(x_1-x_2-x_3)+\alpha(B_2-3)x_1+(\alpha-1)(B_1+2x_1+2x_2+2x_3)+C_0]$
  $+[\alpha(1-\alpha)(x_1+x_2+x_3)+\alpha(B_1+B_2-3)x_1+C_0]-[\alpha(\alpha-1)+$
  $+\alpha(B_2-3)+D]\,(2\alpha x_2x_3+B_0-x_1x_2-x_1x_3-x_2x_3)+$
  $+[\alpha^2+(\alpha+1)(B_2-3)+D+\alpha]\{[\alpha(\alpha-4)+\alpha B_2+D]x_1+$
  $+2[\alpha(\alpha-1)(x_1-x_2-x_3)+\alpha(B_2-3)x_1+$
  $+(\alpha-1)\,(B_1+2x_1+2x_2+2x_3)+C_0]\}x_1=0$
- $2B_2-C_1+D=6$

- $n^2+(2\beta+B_2-4)\,n+\beta(\beta-1)+\beta(B_2-3)+\beta(\beta-1)+\beta(B_2-3)+D=0$
- $B_2x_1^{\,2}-B_1\,x_1+B_0=0$ **(1b-u)**
- $[\beta(\beta-1)(x_2-x_3-x_1)+\beta(B_2-3)x_2+(\beta-1)(B_1+2x_1+2x_2+2x_3)+C_0]\cdot$
  $\cdot[\beta(\beta-1)(x_2+x_3+x_1)+\beta(B_1+B_2-3)x_2+C_0]-[\beta(\beta-1)+$
  $\beta(B_2-3)+D](2\beta x_3x_1+B_0-x_1x_2-x_1x_3-x_2x_3)+[\beta^2+$
  $(\beta+1)(B_2-3)+D+\beta]\{[\beta(\beta-4)+\beta B_2+D]x_1+$
  $2[\beta(\beta-1)(x_2-x_3-x_1)+\beta(B_2-3)x_2+$
  $+(\beta-1)(B_1+2x_1+2x_2+2x_3)+C_0]\}\,x_1=0$
- $2B_2-C_1+D=6$

- $n^2 + (2\gamma + B_2 - 4)\, n + \gamma(\gamma-1) + \gamma(B_2-3) + D = 0$
- $B_2 x_1^{\ 2} - B_1 x_1 + B_0 = 0$ **(1c-u)**
- $[\gamma(\gamma-1)(x_3-x_2-x_1) + \gamma(B_2-3)x_3+(\gamma-1)(B_1+2x_1+2x_2+2x_3) + C_0] \cdot$
  $\cdot [\gamma(1-\gamma)(x_3+x_2+x_1) + \gamma\,(B_1+B_2-3)x_3+C_0] - [\gamma(\gamma-1) +$
  $\gamma(B_2-3)+ D](2\gamma x_1x_2+ B_0 - x_1x_2 - x_1x_3 - x_2x_3) +$
  $[\gamma^2 + (\gamma+1)\,(B_2-3) + D + \gamma]\{[\gamma(\gamma-4) + \gamma B_2 + D]x_1 +$
  $2[\gamma(\gamma-1)(x_3-x_2-x_1) + \gamma(B_2-3)x_3+$
  $+ (\gamma-1)(B_1 + 2x_1+2x_2+2x_3) + C_0\,]\}\, x_1 = 0$
- $2B_2 - C_1 + D = 6$

- $n^2 + (2\alpha+2\beta +B_2-4)\, n +(\alpha+\beta)^2 + (\alpha+\beta)(B_2-4)+D = 0$
- $B_2\, x_1^{\ 2} + B_1\, x_1 + 2\alpha(x_1^{\ 2} -x_1x_2+x_2x_3-x_1x_3) + B_0= 0$ **(1d-u)**
- $[(\alpha^2 - \beta^2)(x_1-x_2) - (\alpha+\beta)^2x_3 + (\alpha+\beta)(x_1+x_2+3x_3+B_1) +$
  $+ (B_2-3)(\alpha x_1+\beta x_2) + 2\alpha x_2+ 2\beta x_1 - B_1-2x_1-2x_2-2x_3 + C_0]\cdot$
  $\cdot [(\alpha^2 - \beta^2)(x_1-x_2) - (\alpha+\beta)^2x_3 + (\alpha+\beta)(x_1+x_2+x_3+B_1) +$
  $+ (B_2-3)(\alpha x_1+\beta x_2)] - [(\alpha+\beta)^2 +(\alpha+\beta)(B_2-4)+D]+$
  $+ (2\alpha x_2x_3 + 2\beta x_3x_1 + B_0-x_1x_2-x_1x_3-x_2x_3) + [(\alpha+\beta)^2 +$
  $+ (\alpha+\beta+1)B_2+ D-2\alpha-2\beta-3]\,\{[(\alpha+\beta)^2 +(\alpha+\beta)(B_2-4)+D]x_1+$
  $+ 2[(\alpha^2-\beta^2)(x_1-x_2) - (\alpha+\beta)^2x_3 + (\alpha+\beta)(x_1+x_2+3x_3+B_1) +$
  $+ (B_2-3)(\alpha x_1+\beta x_2) +2\alpha x_2 +2\beta x_1 - B_1-2x_1-2x_2-2x_3 +C_0]\}x_1 =0$
- $2B_2 - C_1 + D = 6$

- $n^2 + (2\alpha+2\gamma+B_2 -4)\, n +(\alpha+\gamma)^2 +(\alpha+\gamma)(B_2-4)+D = 0$
- $B_2\, x_1^{\ 2} + B_1\, x_1 + 2\alpha(x_1^{\ 2} -x_1x_2+x_2x_3-x_1x_3) + B_0= 0$ **(1e-u)**
- $[(\alpha^2 - \gamma^2)(x_1-x_3) - (\alpha+\gamma)^2x_2 + (\alpha+\gamma)(x_1+3x_2+x_3+ B_1) +$
  $+ (B_2-3)(\alpha x_1+\gamma x_3)+ 2\alpha x_3 +2\gamma x_1 - B_1-2x_1-2x_2-2x_3 +C_0]\cdot$
  $\cdot [(\alpha^2 - \gamma^2)(x_1-x_3) - (\alpha+\gamma)^2x_2 + (\alpha+\gamma)(x_1+x_2+x_3-B_1)+$
  $+ (B_2-3)(\alpha x_1+\gamma x_3)] - [(\alpha+\gamma)^2 + (\alpha+\gamma)(B_2-4) +D] +$
  $+ (2\alpha x_2x_3 + 2\gamma x_1x_2 + B_0 -x_1x_2 -x_1x_3-x_2x_3) + [(\alpha+\gamma)^2 +$
  $+ (\alpha+\gamma+1)B_2+ D- 2\alpha-2\gamma -3]\{[(\alpha+\gamma)^2 +(\alpha+\gamma)(B_2-4)+D]x_1 +$
  $+ 2[(\alpha^2 - \gamma^2)(x_1-x_3)-(\alpha+\gamma)^2x_2+ (\alpha+\gamma)(x_1+3x_2+x_3+ B_1) +$
  $+ (B_2-3)(\alpha x_1+\gamma x_3)+2\alpha x_3 +2\gamma x_1 -B_1 -2x_1-2x_2-2x_3 + C_0]\}x_1 = 0$
- $2B_2 - C_1 + D = 6$

- $n^2 + (B_2 - 4)\, n + (\beta+\gamma)^2 + (\beta+\gamma)(B_2-4) + D = 0$
- $B_2 x_1^{\ 2} - B_1 x_1 + B_0 = 0$ **(1f-u)**
- $[(\beta^2-\gamma^2)(x_2-x_3) - (\beta+\gamma)^2 x_1 + (\beta+\gamma)(x_2+x_3+3x_1+B_1) +$
  $+ (B_2-3)(\beta x_2+\gamma x_3) + 2\beta x_3 + 2\gamma x_2 - B_1 - 2x_1 - 2x_2 - 2x_3 + C_0]\cdot$
  $\cdot [(\beta^2-\gamma^2)(x_2-x_3) - (\beta+\gamma)^2 x_1 + (\beta+\gamma)(x_1+x_2+x_3-B_1) +$
  $+ (B_2-3)(\beta x_2+\gamma x_3)] - [(\beta+\gamma)^2 + (\beta+\gamma)(B_2-4)+D]\cdot$
  $\cdot (2\beta x_1 x_3 + 2\gamma x_1 x_2 + B_0 - x_1 x_2 - x_1 x_3 - x_2 x_3) + [(\beta+\gamma)^2 +$
  $+ (\beta+\gamma+1)B_2 + D - 2\beta - 2\gamma - 3]\ \{[(\beta+\gamma)^2 + (\beta+\gamma)(B_2-4) + D]\, x_1 +$
  $+ 2[(\beta^2-\gamma^2)(x_2-x_3) - (\beta+\gamma)^2 x_1 + (\beta+\gamma)(x_2+x_3+3x_1+B_1) +$
  $+ (B_2-3)(\beta x_2+\gamma x_3) + 2\beta x_3 + 2\gamma x_2 - B_1 - 2x_1 - 2x_2 - 2x_3 + C_0]\}x_1 = 0$
- $2B_2 - C_1 + D = 6$

- $n^2 + (2\beta+2\gamma+B_2-4)n + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma + \alpha(\alpha-1) +$
  $+ \beta(\beta-1) + \gamma(\gamma-1) + (\alpha+\beta+\gamma)b_2 + D = 0$
- $B_2 x_1^{\ 2} + B_1 x_1 + 2\alpha(x_1^{\ 2} - x_1 x_2 + x_2 x_3 - x_1 x_3) + B_0 = 0$ **(1g-u)**
- $[\alpha(\alpha-1)(x_1-3x_2-3x_3) + \beta(\beta-1)(x_2-3x_3-3x_1) + \gamma(\gamma-1)(x_3-3x_2-3x_1) +$
  $+ \alpha\,(B_2 x_1 + B_1 - x_1 + 2x_2 + 2x_3) + \beta(B_2 x_2 + B_1 + 2x_1 - x_2 + 2x_3) +$
  $+ \gamma\,(B_2 x_3 + B_1 + 2x_1 + 2x_2 - x_3) + C_0]\ [\alpha(\alpha-1)(x_1-x_2-x_3) +$
  $+ \beta(\beta-1)(x_2-x_3-x_1) + \gamma(\gamma-1)(x_3-x_2-x_1) + (B_2-3)(\alpha x_1 + \beta x_2 + \gamma x_3) +$
  $(B_1+2x_1+2x_2+2x_3)(\alpha+\beta+\gamma-1) + C_0] - [2\alpha\beta+2\alpha\gamma+2\beta\gamma + \alpha(\alpha-1) +$
  $\beta(\beta-1)+\gamma(\gamma-1) + (\alpha+\beta+\gamma)(B_2-3)+D]\ (2\alpha x_2 x_3 +$
  $2\beta x_3 x_1 + 2\gamma x_2 x_1 + B_0 - x_1 x_2 - x_1 x_3 - x_2 x_3) + [2\alpha\beta+2\alpha\gamma+2\beta\gamma+$
  $\alpha^2+\beta^2+\gamma^2 + (B_2-3)(\alpha+\beta+\gamma+1) + D+\alpha+\beta+\gamma]\{[2\alpha\beta+2\alpha\gamma+2\beta\gamma+$
  $\alpha(\alpha-4) + \beta(\beta-4)+\gamma(\gamma-4)+(\alpha+\beta+\gamma)B_2 + D]x_1 +$
  $2[\alpha(\alpha-1)(x_1-3x_2-3x_3) + \beta(\beta-1)(x_2-3x_3-3x_1) +$
  $\gamma(\gamma-1)(x_3-3x_2-3x_1) + \alpha(B_2 x_1 + B_1 - x_1 +$
  $+ 2x_2 + 2x_3) + \beta(B_2 x_2 + B_1 + 2x_1 - x_2 + 2x_3) +$
  $+ \gamma(B_2 x_3 + B_1 + 2x_1 + 2x_2 - x_3) + C_0\}\ x_1 = 0$
- $2B_2 - C_1 + D = 6$

According to *Theorem* 1, the two particular solutions of each of the equations (1a) - (1g), are given with the formula :

$$z_{1(i)} = e^{-F_i}\left[(x+K_i)(x-x_2)^{n-1}(x-x_3)^{n-1}e^{F_i}\right]^{(n-1)} \qquad (8)$$

$$z_{2(i)} = e^{-F_i}\left[(x+K_i)(x-x_2)^{n-1}(x-x_3)^{n-1}e^{F_i}\cdot\int (x-x_1)^{n-1}(x-x_2)^{1-n}(x-x_3)^{1-n}(x+K_i)^{-2}dx\right]^{(n-1)}$$

where

$$F_i = \int \frac{M_i x + N_i}{(x-x_1)(x-x_2)} dx, \qquad \text{for } i = a, b, c, d, e, f, g \qquad (9)$$

$$M_i = B_{2i} - 4, \quad N_i = B_{1i} + 2(x_2 + x_3) + x_1 B_{2i}, \quad K_i = -\frac{x_1 D_i + n(C_{1i} x_1 + D_i)}{D_i}$$

B_{1a} = - (2αx_3+2αx_2-B_1), B_{2a} = 2α+B_2
C_{1a} = α(α-1) + 4α +α(B_2-3) +C_1, D_a = α(α-1) + α(B_2-3) + D

B_{1b} = - (2βx_1+2βx_3 - B_1), B_{2b} = 2β+B_2
C_{1b} = β(β-1)+ 4β+ β(B_2-3)+C_1, D_b = β(β-1) + β(B_2-3) + D

B_{1c} = - (2γx_2+2γx_1-B_1), B_{2c} = 2γ+B_2
C_{1c} = γ(γ-1) + 4γ + γ(B_2-3) + C_1, D_c = γ(γ-1) + γ(B_2-3) + D

B_{1d} = - (2αx_3+2αx_2+2βx_1+2βx_3 -B_1), B_{2d} = 2α+2β+B_2
C_{1d} = B_2 (α+β) + $(α+β)^2$ + C_1, D_d = $(α+β)^2$ + (α+β)(B_2-4) + D

B_{1e} = - (2αx_3+2αx_2+2γx_1+2γx_2 -B_1), B_{2e} = 2α+2γ+B_2
C_{1e} = B_2 (α+γ) + $(α+γ)^2$ + C_1, D_e = $(α+γ)^2$ + (α+γ)(B_2-4) + D

B_{1f} = - (2βx_1+2βx_3+2γx_1+2γx_2 -B_1), B_{2f} = 2β+2γ+B_2
C_{1f} = B_2 (β+γ) + $(β+γ)^2$ + C_1 , D_f = $(β+γ)^2$ + (β+γ)(B_2-4) + D

B_{1g} = - (2αx_3+2αx_2+2βx_1+2βx_3+2γx_1+2γx_2 -B_1)
B_{2g} = 2α+2β+2γ+B_2
C_{1g} = $α^2$ + $β^2$ + $γ^2$+ 2αβ+2αγ+2βγ+ (α+β+γ)B_2+C_1
D_g = 2αβ+2αγ+2βγ+ α(α-1)+ β(β-1)+γ(γ-1)+(α+β+γ)b_2+D

$$\alpha = \frac{(2-B_2)x_1^{\;2} - (B_1 + 2x_2 + 2x_3)x_1 - B_0 + 2x_2x_3}{(x_3 - x_1)(x_2 - x_1)}$$

$$\beta = \frac{(2-B_2)x_2^{\;2} - (B_1 + 2x_1 + 2x_3)x_2 - B_0 + 2x_1x_3}{(x_1 - x_2)(x_3 - x_2)}$$

$$\gamma = \frac{(2-B_2)x_3^{\;2} - (B_1 + 2x_1 + 2x_2)x_3 - B_0 + 2x_1x_2}{(x_1 - x_3)(x_2 - x_3)}$$

Theorem 2*:* If there is a natural number *n* such that one of the groups of conditions (1a-u) – (1g-u) for the differential equation (1) is satisfied , then this equation is solvable. From the formula (8) and the connecting relations (7), we get two particular solutions of its fundamental system, while the third one can be obtained using the classical method.

Example 1*:* We consider the differential equation

$$(x-1)(x+1)(x-3)\, y'' - (3x^2-6x-1)\, y' + 4(x-2)\, y = 0$$

then $x_1 = 1$, $x_2 = -1$, $x_3 = 3$, $b_0 = 1$, $b_2 = -3$, $b_1 = 6$, $c_1 = 4$, $c_0 = -8$.

The conditions for the existence of one polynomial solution of order 2 for this equation are satisfied, so two particular solutions of its general solution can be found:

$$y_1 = (x+1)^2 , \quad y_2 = \frac{(x+1)^2}{16} \ln\frac{x-3}{x+1} + \frac{x+3}{4} + \frac{x-3}{x+1}$$

They are also particular solutions to the following third order differential equation:

$$(x-1)(x+1)(x-3)\, y''' - 2(x+1)y' + 4y = 0$$

From here, we can find the solutions of each of the transformed equations:

$$(x-1)(x+1)(x-3)\, z_1''' + (4x^2 - 8x - 12)\, z_1'' + (2x - 6)\, z_1' = 0$$

$$z_{11} = \left(\frac{x+1}{x-1}\right)^2 , \quad z_{12} = \left(\frac{x+1}{4x-4}\right)^2 \ln\frac{x-3}{x+1} + \frac{x^2+8x-9}{4(x+1)(x-1)^2}$$

$$(x-1)(x+1)(x-3)\, z_2''' + (4x^2 - 16x + 12)\, z_2'' + (2x - 10)\, z_2' = 0$$

$$z_{21} = 1 , \quad z_{22} = \frac{1}{16} \ln\frac{x-3}{x+1} + \frac{x^2+8x-9}{4(x+1)^3}$$

$$(x-1)(x+1)(x-3)\, z_3''' + (4x^2 - 2)\, z_3'' + (2x - 2)\, z_3' = 0$$

$$z_{31} = \left(\frac{x+1}{x-3}\right)^2 , \quad z_{32} = \left(\frac{x+1}{4x-12}\right)^2 \ln\frac{x-3}{x+1} + \frac{x^2+8x-9}{4(x+1)(x-3)^2}$$

$$(x-1)(x+1)(x-3)\, z_4''' + (8x^2-24)\, z_4'' + (14x-14)\, z_4' + 4\, z_4 = 0$$

$$z_{41} = \frac{1}{(x-1)^2}, \quad z_{42} = \frac{1}{16(x-1)^2}\ln\frac{x-3}{x+1} + \frac{x^2+8x-9}{4(x+1)^3(x-1)^2}$$

$$(x-1)(x+1)(x-3)\, z_5''' + (8x^2-8x-16)\, z_5'' + (14x+2)\, z_5' + 4\, z_5 = 0$$

$$z_{51} = \left(\frac{x+1}{x^2-4x+3}\right)^2, \quad z_{52} = \left(\frac{x+1}{4(x^2-4x+3)}\right)^2 \ln\frac{x-3}{x+1} + \frac{x^2+8x-9}{4(x+1)(x^2-4x+3)^2}$$

$$(x-1)(x+1)(x-3)\, z_6''' + (8x^2-16x+8)\, z_6'' + (14x-18)\, z_6' + 4\, z_6 = 0$$

$$z_{61} = \frac{1}{(x-3)^2}, \quad y_{62} = \frac{1}{16(x-3)^2}\ln\frac{x-3}{x+1} + \frac{x^2+8x-9}{4(x+1)^3(x-3)^2}$$

$$(x-1)(x+1)(x-3)\, z_7''' + (12x^2-24x-4)\, z_7'' + (34x-38)\, z_7' + 16\, z_7 = 0$$

$$z_{71} = \left(\frac{1}{x^2-4x+3}\right)^2, \quad z_{72} = \left(\frac{1}{4(x^2-4x+3)}\right)^2 \ln\frac{x-3}{x+1} + \frac{x^2+8x-9}{4(x+1)^3(x^2-4x+3)^2}$$

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