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Mathematical competitions in Bulgaria -- development and perspectives

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We discuss some aspects of the current situation with the mathematical competition for secondary school students in Bulgaria and with Bulgarian participation. Our analysis is more detailed for our competitions but we address also some important international competitions, including the International Mathematical Olympiad (IMO) and the Balkan Mathematica Olympiad (BMO).

1. Competitions in Bulgaria

The Bulgarian system of mathematical competitions is well developed and enjoys rich traditions with roots back in the beginning of the 70s of 20th century. The system is continuously developing and enriching itself addressing hard problems, most of them to be discussed below.

1.1. Short historical remarks. There were two main types of mathematical competitions in Bulgaria before 1990: national -- Winter mathematical competition (WMC) and Spring mathematical competition (SMC) organized in distinct towns across Bulgaria, and the four rounds of the National olympiad (school, regional, national and selective). The system followed the Soviet model with corresponding adjustments and was well suited with the existing conditions to become great source of ideas, well educated persons and, not at last, successes.

The changes naturally caused some shocks mainly connected with the needs of new ways for organization, management, financing and participation. Fortunately enough, the mathematical society possessed simultaneously potential and reasonable conservatism to soften the effect of the transition crisis.

On the other hand, the new conditions after 1990 created possibilities for new methods which gave (at least in the beginning) very good results. The inevitable problems of the transition were solved ad hoc but some of the are still existing. Meantime, new problems arose mainly connected to the common decrease in the quality of the secondary education.

We consider the main types of mathematical competitions in Bulgaria.

1.2. Test competitions. There are many test (answer to be chosen) competitions either on regional or national level. The easier management and the great interest from the students (unfortunately, caused mainly by the assumption that problems of this kind are easier) destine this variety and there is no reason to expect changes.

The main competitions at national level are "Ivan Salabashev", "Chernorizec Chrabar" and "European Kangaroo". The first two have Bulgarian origin. They started as small competitions and developed with good problems and management. The third is international and the problems are common for the different countries which is delicate because of the differences between the educational systems and the different time to be held. All three competitions issue books with the proposed problems (see for example [5, 6]) which is very helpful for the participants and the teachers. The competitions are organized locally but the national juries prepare national rankings and ensure national awards.

There are some other successful competitions as Christmas', Easter's, and Sofia's mathematical competitions which are aimed to broader public. There are also many reginal competitions where it is worth to mention "St. Nicholas" (Bourgas), "Acad. K. Popov" (Shumen), "Sly Peter" (Gabrovo), "Dimo Maleshkov" and "Rumen Grozdanov" (Plovdiv). The site [7] and the annual calendar for the competitions in 2012-13 give good picture of the number and specifics of these and other competitions.

On the other hand, the well known negative effects of the unduly use of tests appeared also in Bulgaria. Many students become oriented to "strategies" for guessing answers and therefore they fall behind with the analytic thinking and the abilities to find and expose proofs. This was clearly seen in the so-called test-competition for grade 7., which was followed in 2009 by an inappropriate difficult theme proposed for entry high school exam. The results led to the paradox that arbitrary answering (without reading the problems, for instance always answer A, without open answers) give satisfactory mark (more than 3.00). This situation as analyzed in [2]. This absurd helped for solving the problem and not the test-competition for grade 7. does not exist and the entry high school exams are at good level.

1.3. National competitions. The basics of the national competitions go back to 1960-70 but the decisive impetus for their development came with the work of researchers from IMI-BAS and FMI-SU (P. Kenderov, S. Dodunekov and others) in the 70's. There are two such competitions which are held annually in the ends of January and March, respectively. The Autumn competition is very similar in problems, organization and participation.

The national competitions often propose problems at high level of difficulty and originality. The book [1] gives good examples for the quality of the themes. We give the most difficult problems from 2013's editions.

Problem 9.4. Let p is an odd prime. Are there exist positive integers $a, b_1, b_2, \ldots, b_6 \in \{1, 2, \ldots, p-2\}$ such that

$$\binom{p-1}{a}\binom{p-1}{a+1} = \binom{p-1}{b_1}^2 + \binom{p-1}{b_2}^2 + \cdots + \binom{p-1}{b_6}^2 \ ?$$

Problem 11.4. $2n$ points in the plane are given no three on a line. Some of the points are connected with segments in such a way that for every n points there are another point which is connected to all of them. Find the minimum possible number of the segments.

Problem 12.3. Prove that if $a, b > -1,$ then

$$\frac{1+a^6}{1+a} \cdot \frac{1+b^6}{1+b} \geq \frac{1+ab}{2} \cdot \frac{1+a^4b^4}{2}.$$

Investigate the case of equality.

The management and administration of the national competitions however has weak points. One of the recent problems is caused by the prohibition by Ministry of education, science and youth (MESY) to organizers to collect participation fees. Leaving aside the regimentation which reminds the socialist's times this intervention leads to surcharge of budget money (less known competition in different areas appeared) and distorts the competition (since it gives money without clear criteria for quality). It will be simpler and cheaper if the state has clear statement toward natural sciences with special priority of the mathematics (such priorities were explicitly defined by the Presidents Obama and Putin, for USA and Russia, respectively). Then MESY can subsidy the participants, not the organizers, who will fund the competition from the participation fee (according to the interest from the students and teachers).

Another problem is caused by the way of determining stipend. Here the stipend is consequence of the award in a national competition which is quite impractical (and, contrary to the first assumption, non-transparent). Every year we are guilty of giving many places which mean stipends. The solution is obvious and applied everywhere -- announce fixed number of stipends and invite candidates, then evaluate them by independent commission (even by computer).

1.4. National olympiad. The National Mathematical Olympiad in Bulgaria is one of the oldest in Europe. We completed its 62nd edition this year. After 1990 it is organized in three rounds (municipal, district and national) as the first two are at the regions and for the third round all students get together (usually in Sofia). The problems and the organization of the municipal round is responsibility of the corresponding educational inspectorates, for the district round the problems are proposed by the national committee (NC) but the managements is still with the inspectorates. For the national round the problems are prepared by the NC and for the organization is responsible the hosting inspectorate.

The national round is well known around the world as reliable source of good and original problems. This is well documented with many citations of our problems and inclusions in olympic handbooks. In particular there are great interest in English translation (see, e.g. the book [1]). Our problems are widely used in training sessions of other teams (for example Russia, Sweden, Kazakhstan) and sometimes they appear secondly in other competitions. The great international respect for our national round is proved also by the regular participation of the Russian team (in fact, this participation is on exchange basis and will be discussed below).

The problems of the national round follow the style of IMO -- two days, three problems each day, ordered with increasing difficulty according to the NC's opinion. We propose several problems of different difficulty.

Problem 3, 2009. Planes through the points with integer coordinates in the three dimensional Euclidean space partition the space into unit cubes. Find all triples (a,b,c), $a \le b \le c$, of positive integers such that the cubes can be colored in abc colors in such a way that every parallelepiped of dimensions $a \times b \times c$, integer vertices and faces parallel to the coordinate planes does not contain cubes of the same color.

Problem 4, 2009. Let $n \ge 3$ be a positive integer. Find all non-constant real polynomials $f_1(x), f_2(x), \dots, f_n(x)$ such that

$$f_k(x) f_{k+1}(x) = f_{k+1}(f_{k+2}(x)), 1 \le k \le n$$

for every real x (here $f_{n+1}(x) \equiv f_1(x)$ and $f_{n+2}(x) \equiv f_2(x)$).

Problem 3, 2013. All integer points in the plane are colored in three colors. Find the minimum possible positive number S with the following property: for arbitrary coloring there exists a triangle of area S with all three vertices in the same color.

Problem 3, 2012. Let a be a real number distinct from 0 and 1. Alexander and Denitsa play the following game. Alexander moves first and then they alternate, replacing at every move a star with an integral power of a in the equation $\star x^4 + \star x^3 + \star x^2 + \star x + \star = 0$. Alexander wins if the final equation does not have real solutions and otherwise Denitsa wins. Who has a winning strategy?

Problem 6, 2011. In the interior of a convex 2011-gon M 2011 points are chosen in such a way that no three of all 4022 point (the vertices of M and the interior 2011) does not belong to the same line. A coloring of the points in two colors is called good if some of the points can be connected by segments in such a way that:

(1) Every segment connects points of the same color.

(2) Any two segments can intersect only in their ends.

(3) For every two points of the same color there are path consisting of segments.

Find the number of the good colorings.

Problem 2, 2009. The incircle of ΔABC has center I and touches the sides BC, AC and AB at points A_1, B_1 and C_1, respectively. An arbitrary line ℓ through I is given and the points A', B' and C' are symmetric to A_1, B_1 and C_1, respectively, with respect to ℓ. Prove that the lines AA', BB' and CC' are concurrent.

Problem 3, 1997. Let m and n be positive integers and $m+i=a_ib_i^2$ for $i=1,2,\ldots,n$, where a_i and b_i are positive integers and a_i are squarefree. Find all values of n for which there exists m such that $a_1+a_2+\cdots+a_n=12$.

Problem 6, 1998. The sides and diagonals of a regular n-gon X are colored in k colors in such a way that:

(1) For every color a and every segment AB in color a there exists a vertex C of X such that AC and BC are also colored in a;

(2) The three sides of any triangle with vertices among the vertices of X are colored in at most two colors (of course, the intersection points are meaningless).

Prove that $k\le 2$.

The Bulgarian national round is specific with the high technical difficulties (apart from the difficult ideas). The positive and the negative effects of this are discussed every year. We only mention that the most difficult olympiad in technicalities is the China olympiad and this seems to be natural.

Similarly to the case with the national competitions some undone things in the administration of the district and the national round cause problems. In 2012 the fail of an inspectorate to apply adequately the regulations led to pressure on the NC by misleading complaints (trivially, only a part of the regulations was cited, even part of a sentence -- the other part contained the explanation) to MUSE. In fact, the roots of the problem are in the old regulations of MUSE which try to cover all olympiads under the same rules.

The solution of these and other problems (for the national competitions and the olympiad) is found long ago in the most western countries -- MUSE gives a license to an authority to organize the competitions (at present only the Union of the Bulgarian Mathematicians seems to be capable to win such a license). Then MUSE will have only control but not operative functions.

2. Other national olympiads

There are many ways to organize national olympiads. In the most western countries they are assigned to respected professional bodies with corresponding experience. On the east, main role is played by the state structures (usually ministries of education). We consider here the All-Russian olympiad because of the fore mentioned cooperation with the Russian colleagues and the leading world role of this olympiad.

The All-Russian olympiad is usually organized in the second half of April. The students compete according to their grades as the candidates for the national team must participate in grade 11. The competition takes place in two days, with 4 problems each day, ordered with increasing difficulty according to the jury's opinion. The jury consists of more than 30 specialists from different towns of Russia (but mainly from Moscow and St Petersburg) and regularly includes university students (former participants). Many of the problems from the All-Russian olympiad become source of ideas and are used in the training sessions of other teams (including our training).

The teams of Bulgaria and China are regular guest-participants in the All-Russian olympiad. In both cases the participation is on exchange basis. In particular, the regular meetings with the colleagues from China gives opportunity to exchange literature and important information for the experience, traditions and the problems.

The participation of the Bulgarian teams become tradition from 2004. Our students' achievements vary over the years as the three full scores of V. Barzov (2001), R. Kralev (2004) and N. Beluhov (2006) are remarkable.

We give some problems from the All-Russian olympiad in 2010 (the solutions can be found in [3]).

Problem 10.7. (V. Senderov) Given are $n \geq 3$ pairwise relatively prime positive integers. It is known that dividing the product of any $n-1$ of these integers by the remaining one we obtain one and the same remainder r. Prove that $r \leq n$.

Problem 11.3. (P. Kozhevnikov) The quadrilateral $ABCD$ is inscribed in a circle ω and its diagonals intersect in a point K. Points M_1, M_2, M_3 and M_4 are midpoints of the arcs AB, BC, CD and DA (without other vertices of the quadrilateral) respectively. Finally the points I_1, I_2, I_3 and I_4 are incenters for ABK, BCK, CDK and DAK. respectively. Prove that the lines M_1I_1, M_2I_2, M_3I_3 and M_4I_4 are concurrent.

Problem 11.8. (D. Fon-Der Flaas)

In a summer school 512 students follow 9 subjects. All students are arranged in 256 rooms by two students in a room (the two students in a room are called neighbors). It is known that for any two students the subjects that are interesting to both of the are not one and the same (in particular there exists exactly one student with no interesting subjects). Prove that all students can be arranged in a circle in the following way. Any two neighbors are situated next to each other and if two students are next to each other and are not neighbors then the interesting subjects for one of them are exactly those that are interesting to the other one and one more.

It is interesting to mention that the relationships between mathematicians and Ministry officials in Russia are not smooth. For example, the imposed quota on the participants from distinct regions provoked an address to then president of Russia mister Medvedev. After that the misunderstanding has been settled in favor of mathematical community.

3. International competitions

The most important for Bulgaria international competitions are International Mathematical Olympiad (IMO) and Balkan Mathematical Olympiad (BMO).

3.1 International Mathematical Olympiad. Bulgaria is among just three countries (together with Rumania and Hungary), that have participated in all IMO's. This fact, the excellent achievements of our students and the prestige of Bulgarian National Olympiad forces all people involved to work for development of the olympiad.

Bulgarian problems are often included in the short-list and some of them are chosen in the papers for the competition. The last two problems included are given below (author of both of them is A. Ivanov).

Problem 6, 2001. Let $a>b>c>d$ be positive integers such that $ac+bd=(b+d+a-c)(b+d-a+c)$. Prove that $ab+cd$ is composite.

Problem 2, 2003. Find all pairs of positive integers a and b such that $\dfrac{a^2}{2ab^2-b^3+1}$ is an integer.

A book [4] with all short-listed problems and their solutions from IMO have become invaluable source for preparation of all teams. The official cite of IMO [8] provides comprehensive information for all IMO's, problems, medals, hall of fame, etc. We present here some of the most intriguing problems from the recent years.

Problem 3, 2012. (Canada) The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players. At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful. After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \ge 2^k$, then B can guarantee a win.

2. For all sufficiently large k, there exists an integer $n \ge 1.99^k$ such that B cannot guarantee a win.

Problem 6, 2011. Let ABC be an acute triangle with circumcircle Γ. Let all be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ell in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Problem 6, 2009. Let $a_1, a_2, \dots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \dots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

Problem 3, 2007. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Problem 2, 2011. (Great Britain) Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A windmill is a process that starts with a line l going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line l going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

Similar to the national competitions the main conflict are not of mathematical nature. The procedure of choosing the problems for the students involves the preparation of so called short-list (this is done by the problem selection committee appointed by the organizers). Then the leaders of all teams after two or three days deliberation select the problems they prefer. For many people this system needs fundamental change since it allows unfair behavior. The efforts of newly appointed ethic commission are still not enough for major change in the attitude of people looking for easy ways to earn medals.

3.2 Balkan Mathematical Olympiad. BMO is held annually since year 1984 and the intention of the first organizers (the merits go to I. Tonov) as a part of the preparation for the teams for IMO. At present 10 Balkan and 10 invited countries take part in BMO which involves around 120. Bulgaria (together with Rumania and Greece) has participated in all editions of BMO. This year though it is expected that this year BMO will be canceled.

BMO is one day competition and includes four problem in increasing difficulty (at the jury discretion). The popularity and prestige of BMO result in constant rising number of invited countries. Countries like Great Britain, France, Italy and Kazahstan are among regular participants. Although BMO is not of the level of All-Russian olympiad or IMO it offers high quality problems. We present some of the most interesting ones from recent years.

Problem 3, 2011. (Bulgaria, A. Ivanov) Let S be finite set of positive integers with the following property: for any x from S all positive divisors of x are also elements of S. A nonempty subset T of S is called **good** if $x, y \in T$, $x > y$ implies that the ratio x/y is a power of a prime number. A nonempty subset T of S is called **bad** if $x, y \in T$, $x > y$ implies that the ratio x/y is not a power of a prime. All subsets of S with exactly one element are simultaneously good and bad. Let k be the maximum cardinality of a good set. Prove that the minimum possible number of bad sets no two of which intersect and having union the whole S equals k.

Problem 4, 2010. (Turkey) For any positive integer $n \geq 2$ denote by $f(n)$ the sum of all positive integers that are less than n and are not relatively prime with n. Prove that $f(n+p) \neq f(n)$ for any positive integer n and any prime p.

Problem 4, 2009. (Bulgaria, N. Nikolov) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f^2(m)+2f^2(n)) = m^2+2n^2$$

for arbitrary $m,n\in\mathbb{N}$.

problem 4, 2005. (Moldova) Let $n\geq 2$ be an integer. Let S be a subset $\{1,2,\dots n\}$ such that there are no elements from S one of which is a divisor of the other one and there are no two elements of S that are relatively prime. Find the maximum possible number of elements of S.

Problem 4, 2007. (Turkey) Let $n\geq 3$ be positive integer and C_1, C_2 and C_3 be contours of three convex n-gons in the plane. It is known that all intersections $C_1\cap C_2$, $C_2\cap C_3$ and $C_3\cap C_1$ are finite. Find the maximum possible number of points in the set $C_1\cap C_2\cap C_3$.

Problem 2, 2007. (Bulgaria, N. Nikolov) Find all functions $f:\mathbb{R}\to\mathbb{R}$, such that

$$f(f(x)+y) = f(f(x)-y)+4f(x)y \qquad x,y\in\mathbb{R}.$$

The problems and challenges for BMO are similar to those for IMO with the corresponding "balkanization". For the first time since the beginning of BMO political debate is set to cancel BMO 2013. The previous BMO 2012 held in Turkey is to be remembered with three main features. First, the excellent organization by the host country which is a standard for all competitions in Turkey. Second, problem 4 from the paper turned out to be the same as problem given on the National olympiad of USA held just few days before the BMO. The reason was found to be that one of the invited countries, Saudi Arabia, proposed the same problem twice. Third, the rules of the olympiad have been violated by giving 14 gold medals (including 1 for the second team of Turkey and none for the invited countries) whereas the regulations allow maximum of 8.

4. Conclusion

The above analysis shows that the challenges for all mathematical competitions are similar. Those having mathematical nature are more or less easily solved by the people involved. Considerably more difficult is to cope with problems having non mathematical background.

Acknowledgement. The first author is also with Southwestern University, Faculty of Mathematics and Natural Sciences, Department of Informatics (66 Ivan Mihailov str., Blagoevgrad 2700, Bulgaria).

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On categorical semigroups

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Abstract: *The structure of categorical at zero semigroups is studied from the point of view their likeness to categories.*

1. The connection between small categories and semigroups is well-known. Let $\underline{C}$ be a small category. We join an extra element 0 to the set of its morphisms. The obtained set $S(\underline{C})$ becomes a semigroup (with the zero 0) with respect to the operation $*$:

$$f * g = \begin{cases} fg & if\ \ fg\ \ is\ \ defined, \\ 0 & otherwise. \end{cases}$$

Besides, $S(\underline{C})$ satisfies a condition:

if $f * g * h = 0$, then either $f * g = 0$ or $g * h = 0$.

Semigroups with this property are called *categorical at zero*. We shall name them K*-semigroups* for brevity.

So, each small category is a K-semigroup. Of course, the converse is not right. Moreover, there are natural examples of K-semigroups, close to categories, but not being those. Here are two of them:

Example 1 Any family of metric sets and all contracting mappings between them.[1]

Example 2 Any family of sets of the given infinite cardinal number p and all their mappings $f: A \to B$ such, that $|B \backslash fA| = p$.

Both examples do not contain identity morphisms and therefore are not categories.

Here it is worth to mention also quasi-categories of C. Ehresmann [2]. One more example will be considered below.

This note is devoted to the study of the structure of K-semigroups. We hope that it will be useful for understanding what could be "near relatives" of categories.

[1] We are indebted to Prof. S. Favorov for this example.

For categories we use terminology from [5]. Necessary data from the theory of semigroups can be found in [1] and [4].

2. We shall need two classes of semigroups.

Recall that a semigroup S with a zero is called *n-nilpotent* if $S^n = 0$.

Now let G be a group, I and Λ be sets, $W = (w_{\lambda i})_{\lambda\in\Lambda, i\in I}$ a matrix whose elements $w_{\lambda i}$ are taken from the group with a jointed zero $G \cup 0$. The set $(I \times G \times \Lambda) \cup 0$ with the (associative) operation

$$(i, g, \lambda)(j, h, \mu) = (i, g w_{\lambda j} h, \mu)$$

is called *a Rees semigroup* and is denoted by $M = M^0(G;\ I, \Lambda;\ W)$.

We shall use the special case of Rees semigroups, when $G = 1$, and write the non-zero elements of $M = M^0(1;\ I, \Lambda;\ W)$ as (i, λ). Then the multiplication has a form

$$(i, \lambda)(j, \mu) = \begin{cases} (i, \mu) & if \quad w_{\lambda j} = 1, \\ 0 & if \quad w_{\lambda j} = 0. \end{cases}$$

It is easy to see that except small categories, also 2-nilpotent semigroups and Rees semigroups are K-semigroups.[2]

3. Further S will denote a K-semigroup. We call the subset $Ann_l S = \{a \in S | aS = 0\}$ by the *left annihilator* of S; similarly the *right annihilator* is the subset $Ann_r S = \{a \in S | Sa = 0\}$. The union

$$\mathrm{Ann}_q S = \mathrm{Ann}_l S \bigcup \mathrm{Ann}_r S.$$

is called a *quasi-annihilator*. Obviously, both left and right annihilators (and, hence, the quasi-annihilator) are two-sided ideals.

First we consider 3-nilpotent K-semigroups.

Lemma 1 *S is 3-nilpotent if and only if it coincides with $Ann_q S$.*

Proof. Let S is 3-nilpotent and $a \notin \mathrm{Ann}_l S$. Then $ab \neq 0$ for some $b \in S$. Since $xab = 0$ for all $x \in S$, $xa = 0$, i. e. $a \in \mathrm{Ann}_r S$. Hence $S = \mathrm{Ann}_q S$.

Conversely, if $S^3 \neq 0$, then $abc \neq 0$ for some $a, b, c \in S$. Then $ab \neq 0$ and $bc \neq 0$, i. e. $b \notin \mathrm{Ann}_r S \cup \mathrm{Ann}_l S$, whence $S \neq \mathrm{Ann}_q S$. ∎

Corollary 1 *The quasi-annihilator of a K-semigroup is 3-nilpotent.* ∎

Lemma 1 allows to build all 3-nilpotent K-semigroups. Namely, let a set A be given with a fixed element 0, two subsets $B, C \subseteq A$ such, that $A = B \cup C$, $\quad B \cap C \ni 0$, and a mapping $\varphi: (B \backslash C) \times (C \backslash B) \to B \cap C$, which satisfy conditions:

[2]Generally speaking, it is possible to turn every semigroup into a K-semigroup, if we join a zero element to it, but such point of view, certainly, is ineffective.

a) for every $b \in B\backslash C$ there is $c \in C\backslash B$ such that $\varphi(b,c) \neq 0$;
b) for every $c \in C\backslash B$ there is $b \in B\backslash C$ such that $\varphi(b,c) \neq 0$.

Define a multiplication on A:

$$xy = \begin{cases} 0, \ if \quad x \in C \quad or \quad y \in B, \\ \varphi(x,y), \ if \quad x \in B\backslash C \ and \quad y \in C\backslash B. \end{cases}$$

If $xy \neq 0 \neq yz$ then $y \in B \cap C$ and we get the contradiction. Therefore for all $x, y, z \in A$ either $xy = 0$ or $yz = 0$. It follows from here both 3-nilpotency and categoricity at zero. Besides the conditions a) and b) provide equalities $C = \mathrm{Ann}_l A,\ B = \mathrm{Ann}_r A$.

Now it is reasonable to consider the quotient semigroup $S/\mathrm{Ann}_q S$. It turns out that S is a ``splitting'' ideal extension of $Ann_q S$ in the following sense:

Lemma 2 *The subset* $T = (S\backslash Ann_q S) \cup 0$ *is a subsemigroup.*

Proof. Suppose that $a, b \in T$, $ab \in \mathrm{Ann}_q S$ and $ab \neq 0$. Let, e. g., $ab \in \mathrm{Ann}_l S$. Then $abx = 0$ for all $x \in S$, whence $bx = 0$, $b \in \mathrm{Ann}_l S$. ■

Note that $S/\mathrm{Ann}_q S \cong T$.

4. To study the structure of T we introduce the following relations[3] on S:

$$\mathcal{P} = \{(a,b) \in S \times S | \forall x \in S \quad xa = 0 \Leftrightarrow xb = 0\},$$
$$\mathcal{Q} = \{(a,b) \in S \times S | \forall x \in S \quad ax = 0 \Leftrightarrow bx = 0\},$$
$$\mathcal{N} = \mathcal{P} \cap \mathcal{Q}.$$

In what follows we shall consider the restrictions of $\mathcal{P}, \mathcal{Q}$ and $\mathcal{N}$ on T an denote them by the same letters.

Obviously, $\mathcal{P}, \mathcal{Q}$ and $\mathcal{N}$ are equivalences. Furthermore, for each of them 0 forms the single-element class (for example, if $a \in T$, $a\mathcal{P}0$ then $Ta = 0$, i. e. $a \in \mathrm{Ann}_r S \cap T = 0$).

Denote by P_i and Q_λ $\mathcal{P}$- and $\mathcal{Q}$- classes respectively ($i \in I,\ \lambda \in \Lambda$, where I and Λ are sets of indexes). Set $N_{i\lambda} = P_i \cap Q_\lambda$. So defined class $N_{i\lambda}$ is either empty or a $\mathcal{N}$-class. Let

$$P_i^0 = P_i \cup 0, \qquad Q_\lambda^0 = Q_\lambda \cup 0, \qquad N_{i\lambda}^0 = N_{i\lambda} \cup 0.$$

A homomorphism, for which the complete preimage of zero is one-element, is called *0-restricted*. A congruence, corresponding to a 0-restricted homomorphism, we shall name also *0-restricted*.

[3]They were defined by L. Gluskin [3] for 0-simple semigroups.

Lemma 3 *$\mathcal{N}$ is the greatest 0-restricted congruence on T.*

Proof. Let $(a,b) \in \mathcal{N},\ t \in T$. Show that $(ta,tb) \in \mathcal{N}$. If $ta = 0$ then also $tb = 0$ (because $(a,b) \in \mathcal{P}$), i. e. $(ta,tb) \in \mathcal{P}$. Let $ta \neq 0 \neq tb$. If $xta = 0$ then $xt = 0$, whence $xtb = 0$ and $(ta,tb) \in \mathcal{P}$. The same way $(ta,tb) \in \mathcal{Q}$.

So $(ta,tb) \in \mathcal{N}$, similarly $(at,bt) \in \mathcal{N}$; hence, $\mathcal{N}$ is a congruence. Evidently, $\mathcal{N}$ is 0-restricted.[4]

Let ρ be a 0-restricted congruence on T, $(a,b) \in \rho$. If $xa = 0$, then $(0,xb) \in \rho$, whence $xb = 0$. Analogously, $ax = 0$ if and only if $bx = 0$. Hence, $(a,b) \in \mathcal{N}$ and $\mathcal{N}$ is the greatest 0-restricted congruence. ■

Now we elucidate, what is the quotient semigroup $T/\mathcal{N}$:

Lemma 4 $P_i Q_\lambda \subseteq N_{i\lambda}^0$.

Proof. Since P_i^0 is a right ideal and Q_λ^0 is a left one, then $P_i Q_\lambda \subseteq (P_i \cap Q_\lambda) \cup 0 = N_{i\lambda}^0$. ■

Corollary 2 *If $N_{i\lambda}$ and $N_{j\mu}$ are non-empty then $N_{i\lambda} N_{j\mu} \subseteq N_{i\mu}^0$.* ■

Lemma 5 *For every $i \in I,\ \lambda \in \Lambda$ either $0 \notin Q_\lambda P_i$ or $Q_\lambda P_i = 0$.*

Proof. Suppose that $0 \in Q_\lambda P_i$, i. e. $yx = 0$ for some $x \in P_i$, $y \in Q_\lambda$. Choose arbitrary $u \in P_i,\ v \in Q_\lambda$. From $(x,u) \in \mathcal{P}$ it follows $yu = 0$; from here and from $(y,v) \in \mathcal{Q}$ it follows $vu = 0$. Therefore $Q_\lambda P_i = 0$. +

Corollary 3 *Assume that $N_{i\lambda}$ and $N_{j\mu}$ are non-empty. If $Q_\lambda P_j = 0$ then $N_{i\lambda} N_{j\mu} = 0$. If $Q_\lambda P_j \neq 0$ then $N_{i\lambda} N_{j\mu} \subseteq N_{i\mu}$.* ■

In particular, we obtain some information about $\mathcal{N}$-classes:

Corollary 4 *$N_{i\lambda}^0$ is either a semigroup with zero multiplication or a semigroup with an joined extra zero.* ■

Consider a Rees semigroup $M = M^0(1;\ I,\Lambda;\ W)$ with a sandwich matrix $W = (w_{\lambda i})_{\lambda\in\Lambda, i\in I}$, obeyed the condition:

$$w_{\lambda i} = \begin{cases} 1 & if \quad Q_\lambda P_i \neq 0, \\ 0 & if \quad Q_\lambda P_i = 0. \end{cases}$$

[4]Indeed $\mathcal{N}$ is also a congruence on S, but not 0-restricted.

Set $\varphi(0) = 0$ and $\varphi(N_{i\lambda}) = (i, \lambda)$ for every nonempty $\mathcal{N}$-class $N_{i\lambda}$. In that way a mapping $\varphi: T/\mathcal{N} \to M$ is defined. By Corollary 3

$$\varphi(N_{i\lambda}N_{j\mu}) = \varphi(N_{i\mu}) = (i, \mu) = (i, \lambda)(j, \mu) = \varphi(N_{i\lambda})\varphi(N_{j\mu}),$$

if $Q_\lambda P_i \neq 0$, and

$$\varphi(N_{i\lambda}N_{j\mu}) = \varphi(0) = 0 = (i, \lambda)(j, \mu) = \varphi(N_{i\lambda})\varphi(N_{j\mu}),$$

if $Q_\lambda P_i = 0$. Hence, φ is a monomorphism.

So we have proved the following assertion:

Theorem 1 *Every K-semigroup S is a splitting ideal extension of a 3-nilpotent ideal A by the subsemigroup $T = (S \backslash A) \cup 0$. The quotient semigroup of T by its greatest 0-restricted congruence is isomorphic to a subsemigroup of a Rees semigroup with the one-element basic group.* ■

5. The obtained results are interpreted for categories as follows. Let K-semigroup S is a small category. Then $Ann_q S = 0$, i. e. $S = T$. Further, the fact that elements from $\mathcal{P}$-class P_i are annihilated at the left by the same elements, means that P_i is the set of all arrows which end in the object i. Similarly, Q_i is the set of arrows starting out i. From here it follows that I and Λ can be identified each with other and with the set of the objects of the category S. It is easy to see that $Q_j P_i \neq 0$ if and only if $i = j$. Besides, $N_{ij} = \mathrm{M}or(j, i)$, hence $N_{ij}N_{kl} = 0$ for $j \neq k$, and N_{ii} is a monoid. At last, the homomorphism φ is a functor from S to $S/\mathcal{N}$, bijective on the objects.

We finish the article with an example of a ``non-category''.

Example 3 Let $\underline{C}$ be a small category, $\underline{D}$ and $\underline{\Delta}$ its subcategories. For the simplicity we assume that for every object a from $\underline{D}$

$$\mathrm{M}or(a, \underline{\Delta}) = \bigcup_{\alpha \in \mathrm{O}b\underline{\Delta}} \mathrm{M}or(a, \alpha) \neq \emptyset,$$

and choose a morphism $\varepsilon_a \in \mathrm{M}or(a, \underline{\Delta})$ for each $a \in \mathrm{O}b\underline{D}$. Let $\varepsilon_a: a \to \overline{a}$.

Denote

$$\mathrm{M}or(\underline{\Delta}, \underline{D}) = \bigcup_{\alpha \in \mathrm{O}b\underline{\Delta}, a \in \mathrm{O}b\underline{D}} \mathrm{M}or(\alpha, a)$$

and $S = \mathrm{M}or(\underline{\Delta}, \underline{D}) \cup \{0\}$ where 0 is an extra zero element. Define an operation $*$ on S: for $f \in \mathrm{M}or(\alpha, a)$, $g \in \mathrm{M}or(\beta, b)$

$$g * f = \begin{cases} g\varepsilon_a f, & if\ \overline{a} = \beta, \\ 0 & otherwise. \end{cases}$$

It is easy to see that S becomes a K-semigroup. In this case $Ann_r S = 0$ and $Ann_l S = \{g | \mathrm{d}omg \in \mathrm{O}b\underline{\Delta} \backslash \varepsilon(\underline{D})\}$ where $\varepsilon(\underline{D}) = \{\overline{a} | a \in \mathrm{O}b\underline{D}\}$.

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Bootstrap with Cycled Blocks in Stationary Time Series

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Abstract: *Ekonomi and Butka used the bootstrap with cycled blocks to estimate the fractional parameter in ARFIMA model. These blocks are compounded by a fix number of cycles. The number of cycles and their length are random. These blocks are not overlapping and moving. In the paper we have shown some bootstrap schemes with cycled blocks and have proved that these bootstrap estimators are consistent in stationary time series when some conditions are fulfilled.*
Keywords: *bootstrap, stationary, time series, cycling blocks, consistency..*

1. INTRODUCTION

When parametric modelling and theoretical analysis are difficult, the bootstrap (Efron, 1979) is a good alternative for calculating standard errors or constructing confidence regions for parameters. This method use single observations as the resampling units. However, bootstrap schemes for time series data resample sets of consecutive observations to capture the process correlation structure.

Kunsch (1989) and Liu and Singh (1992) independently introduced the moving blocks bootstrap (MBB) which work by randomly resampling overlapping blocks of fixed size. Politis and Romano (1992) introduced a variant of the MBB which amounts to wrapping the data around in a circle before blocking them. Politis and Romano (1994) proposed another alternative, the stationary bootstrap (SB), which works by selecting blocks formed with random starting points and random lengths.

Despite the asymptotic attractiveness of the MBB and SB, they both suffer from a handicap. When blocks are randomly selected and concatenated to construct bootstrap samples, the dependence structure near the block endpoints is incorrect. Kunsch and Carlstein (1990) proposed the linked blockwise bootstrap and Carlstein et al. (1998) proposed the matched-block bootstrap in order to correct this problem.

Ekonomi and Butka (2011) used the bootstrap with cycled blocks to estimate the fractional parameter in ARFIMA model. In this paper, we develop the bootstrap with cycled blocks in stationary time series. The outline of the paper is as follows. In Section 2, we define the bootstrap with blocks compounded by cycles. We have two kinds of blocks: not overlapping and moving blocks. In Section 3, we have studied the proposed bootstrap properties in stationary time series. In Section 4, we have done a discussion about the determination of the best size of a block.

2. BOOTSTRAP ESTIMATION WITH CYCLED BLOCKS

Let us have a time series X_t with terms $X_1, X_2, ..., X_n$. We shall show how the blocks will be formed on the basis of which the bootstrap sample will be constructed. If we subtract from the series terms their mean, the time series will exibit consecutive groups with continuosly alternating positive and negative signs. Let us write the adapted series as $X'_1, X'_2, ..., X'_n$. Every two consecutive groups of positive and then negative values, we will call a cycle. Let $C_1, C_2, ..., C_k$ denote the created cycles and that each cycle C_i has n_i values, for i=1,2,...,k. We have $\sum_{i=1}^{k} n_i = n$. The form of the cycles will be $C_1 = \{X_1, ..., X_{n_1}\}$, $C_2 = \{X_{n_1+1}, ..., X_{n_1+n_2}\}$, ..., $C_k = \{X_{n_1+...+n_{k-1}+1}, ..., X_n\}$, with $X'_{n_1+...+n_i} < 0$ and $X'_{n_1+...+n_i+1} > 0$ for i=1,2,...,k. It is clear enough that the length of each cycle and their number are random.

We define a block costructed of a fixed number of consecutive cycles. We describe two different ways of forming of blocks. In the first case the blocks are not overlapping and, if the block is compounded by s consecutive cycles, the starting point will be the obseravtions X_1, $X_{n_1+...+n_s+1}$, ... In the second case we have the moving blocks and the starting point of a block will be the obseravtions X_1, X_{n_1+1}, ..., $X_{n_1+...+n_k+1}$.

For the bootstrap estimation we randomly choose with a probability P a number of blocks to form a bootstrap sample. We concatenate the choosed blocks and we truncate tha last choosed block, for the bootstrap sample has n observation.

3. THE CONSISTENCY OF THE PROPOSED BOOTSTRAP

Let we explain some special scheme of the bootstrap with cycled blocks. We see that these bootstrap methods are consistent in stationary time series.

3.1. Bootstrap with not-overlapping blocks

Without lose the generalization let suppose that a block is compounded by only one cycle, so s=1. Let S_i for i=1,2,...,k indicate the sum of the elements in the i-th block. We choose from the set $\{(S_i,n_i)\}$, for i=1,2,...,k, with the same probability until we have a subset of n values from the time series. Let be $\{(S_i^*,n_i^*)\}$ the bootstrap sample. Then S_r^*,n_r^* have different distribution with S_i^*,n_i^*, for i=1,2,...,r-1. So

(1) $$\sum_{i=1}^{r} n_i^* = n .$$

We assume that S_i^* is the sum of the observations of the i-th choosed block and this block has n_i^* observations, for i=1,2,…,r-1. It is clear enough that r is a random variable. We mention that the larger the n_i^*'s, the smaller r becomes and vice versa. We can see that n_i^* and S_i^*, for i=1,2,..,r, and r are independent. We have the following bootstrap estimator for the time series mean

(2) $$\overline{X}_n^* = \frac{1}{n}\sum_{i=1}^{n} X_i^* = \frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{n_i^*} X_{ij}^* = \frac{1}{n}\sum_{i=1}^{r} S_i^* \quad .$$

Then

(3) $$E^*\overline{X}_n^* = \frac{1}{n}E^*\sum_{i=1}^{r} S_i^* = \frac{1}{n}\left(E^*(r-1)E^*\left(S_i^*\right)+E^*\left(S_r^*\right)\right) .$$

But

(4) $$E^*\left(S_i^*\right) = \frac{1}{k}\sum_{i=1}^{k} S_i = \frac{n\overline{X}_n}{k}$$

and

(5) $$E^*\left(n_i^*\right) = \frac{1}{k}\sum_{i=1}^{k} n_i = \frac{n}{k}$$

From (1) we have $E^*(r-1)E^*\left(n_i^*\right)+E^*\left(n_r^*\right)=n$ and

(6) $$E^*(r-1) = \frac{k\left(n - E^*\left(n_r^*\right)\right)}{n}$$

Plugging (4) and (6) into (3), we have

(7) $E^{*}\overline{X}_{n}^{*}=\overline{X}_{n}+o_{p}(1)$

since $E^{*}(S_{r}^{*})$ and $E^{*}(n_{r}^{*})$ are bounded. From (7) we see that the bootstrap estimator (2) is consistent.

We can see another bootstrap scheme. This is an unbalanced scheme. Let we denote by $Y_1, Y_2, \ldots, Y_k$ the means of each block. We choose randomly from these values with probability $\frac{n_i}{n}$ with replacement. Let we have the bootstrap sample $Y_1^*, Y_2^*, \ldots, Y_k^*$. Their mean is $\overline{Y}_k^*=\frac{1}{k}\sum_{i=1}^{k}Y_i^*$. We have

(8) $E^{*}\overline{Y}_{k}^{*}=E^{*}Y_{1}^{*}=\frac{n_1}{n}y_1+\frac{n_2}{n}y_2+\ldots\frac{n_k}{n}y_k=\overline{X}_n$

From (8) we see that the mean of this bootstrap estimator is equal with the observed time series mean.

3.2. Bootstrap with moving blocks

Now the blocks are compounded by a fix number of cycles. We can fix s=2. The forming scheme of bootstrap sample is the same with the bootstrap scheme with not overlapping blocks. Let we denote by $D_1, D_2, \ldots, D_{k-1}$ the sum of the blocks observations and let suppose that the blocks have each $l_1, l_2, \ldots, l_{k-1}$ elements. So $D_1=S_1+S_2$, $D_2=S_2+S_3, \ldots$, $D_{k-1}=S_{k-1}+S_k$. We can see that $D_1+D_2+\ldots+D_{k-1}=2(S_1+S_2+\ldots+S_k)-(S_1+S_k)$ and $l_1+l_2+\ldots+l_{k-1} = 2(n_1+n_2+\ldots+ n_k) -(n_1+n_k)$.

We choose from the set $\{(D_i,l_i)\}$, for i=1,2,...,k-1, with the same probability until we have a subset of n values from the time series. Let $\{(D_i^*,l_i^*)\}$ the bootstrap sample. Then D_k^*,l_k^* have different distribution with D_i^*,l_i^* for i=1,2,...,k-1. We have

(9) $\sum_{i=1}^{r}l_i^*=n$.

The bootstrap estimator for the time series mean is $\overline{X}_n^*$ or the mean of the bootstrap sample. So

(10) $E^{*}\overline{X}_{n}^{*}=\frac{1}{n}E^{*}(\sum_{i=1}^{r}D_i^*)=\frac{1}{n}\left(E^{*}(r-1)E^{*}(D_i^*)+E^{*}(D_r^*)\right)$

We have

$$E^{*}(D_i^*)=\frac{1}{k-1}(\sum_{i=1}^{k}D_i)=\frac{1}{k-1}\left(\sum_{i=1}^{k}S_i-(S_1+S_k)\right)=\frac{2n\overline{X}_n}{k-1}-\frac{S_1+S_k}{k-1}$$

and

$$E^*(l_i^*) = \frac{2n}{k-1} - \frac{n_1 + n_k}{k-1}$$

From (9) we can take $E^*(r-1)E^*(l_i^*) + E^*(l_r^*) = n$ or

$$E^*(r-1) = \frac{n - E^*(l_r^*)}{E^*(l_i^*)} = \frac{(n - E^*(l_r^*))(k-1)}{2n - (n_1 + n_k)}$$

After some calculations we have

$$E^*\overline{X}_n^* = \overline{X}_n + \frac{n_1 + n_k - 2E^*l_r^*}{2n - (n_1 + n_k)}\overline{X}_n - \frac{n - E^*l_r^*}{n(2n - (n_1 + n_k))}(S_1 + S_k) + \frac{E^*D_r^*}{n}$$

If the time series X_t is stationary and $E^*\left(n_i^2\right) < \infty$ (Fuller 1996), we have $n_i = O_p(1)$ and $\overline{X}_i = O_p\left(n^{-1/2}\right)$

Then we have $E^*\overline{X}_n^* = \overline{X}_n + o_p(1)$. This expression shows the consistency of the bootstrap estimation.

To avoid the sums of the first and last cycle terms in above expressions, we follow after the time series observations, the observations of the first cycle. So we have $N=n+n_1$ observations. In this situation we have k+1 blocks. So $D_1+D_2+\ldots+D_k=2(S_1+S_2+\ldots+S_k)$ or $l_1+l_2+\ldots+l_k=2n$.

In analogous way, we have

$$E\overline{X}_n^* = \frac{1}{n}\left[E^*(r-1)E^*(D_i^*) + E^*(D_r^*)\right], \qquad ED_i^* = \frac{2n}{k}\overline{X}_n, \qquad El_i^* = \frac{2n}{k}$$

$$E^*(r-1) = \frac{k(n - E^*(l_r^*))}{2n}$$

and finally the expression (7), that show the consistency of the bootstrap estimator.

4. THE DETERMINATION OF THE SIZE OF A BLOCK

Given a performance criterion by which to assess the quality of bootstrap resampling schemes, the resampling units must be determined appropriately, since different choices lead to quite different estimates..

In the case of the MBB, two empirical methods were proposed by Hall et al. (1995) and Lahiri (1996) to determine the block size. Hall et al. (1995) established that the best block size, in the sense of minimum mean squared error of estimate, depends on three factors: the autocorrelation structure of the series, the series length, and the task to be performed. They provided an algorithm to iteratively estimate the best block size, although the results seem to be somewhat erratic due to series-to-series variability.

To our knowledge, an algorithm to determine the optimal block size of the SB has not yet been developed. Ekonomi and Butka (2011) found empirically that the best choice for the size block was 10 for non overlapping blocks and 2, 8, 9, 10 for moving blocks when the time series observations number was n=300. Before we proceeded to the performance comparison, we had to find the optimal block sizes. Since no analytic expressions are available to compute the optimal block sizes, we employed numerical search to determine the optimal block sizes empirically.

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Jackknife With Re-Blocks in Time Series with Weak Dependence

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Abstract: *The jackknife was resulted to be a successful tool for estimating the bias and variance in the case when the observations are with independent and identically distribution. The assumption of independent observations is crucial. In time series, where the phenomena of dependence through the terms is evident, Kunsch (1989) and Liu and Singh (1992) introduced the jackknife with blocks, by deleting blocks of consecutive observations one at a time. The blocks were moved or non-overlapping. In this paper we have proposed a jackknife with blocks formed from re-calculated values of a statistic. We have showed that it works well in the case of strictly stationary α-mixing time series.*

Keywords: *jackknife, variance, time series, re-blocks, weak dependence.*

1. INTRODUCTION

The jackknife (Efron 1979, 1982) has proven to be a powerful tool for approximating the sampling distribution and variance of complicated statistics defines on a sequence of independent identically (i. i. d.) random variables (Bickel and Freedman (1981), Singh (1981). However, the assumption of independence of the observations is crucial. It is easy to see that they give incorrect answers if the dependence is neglected (Singh (1981)). Because of these reasons Kunsch (1989) and Liu and Singh (1988) independently proposed an extension of the standard jackknife which does not require to fit a parametric or semiparametric model first. It worked for arbitrary stationary processes with mixing conditions. For the jackknife we deleted each block of the consecutive observations with equal length once and calculated the sample variance of the values of statistics in this way. The blocks were moving blocks. The idea of dividing the time series in blocks is used many times in the study of time series. We can mention the works of Carlstein (1986), Phillips and Yu (2005), Ekonomi and Butka (2011).

In this paper we propose a jackknife method for estimating the variance of an estimator in a time series strictly stationary α-mixing. In Section 2 we have shown the idea of this jackknife estimation. In Section 3 we have argued the validity of the jackknife method under some conditions.

2. VARIANCE JACKKNIFE ESTIMATION WITH RE-BLOCKS

It is given the time series X_t, $t \in Z$. Suppose that we have the observations $X_1, X_2, \ldots, X_N$ from this time series. We divided these observations in blocks compounded with s consecutive observations in the form $S_1=\{X_1,\ldots,X_s\}$, $S_2=\{X_2,\ldots,X_{s+1}\},\ldots$, $S_{N-s+1}=\{X_{N-s+1},\ldots,X_N\}$. If we denote $n=N-s+1$, we have formed the blocks $S_1, S_2,\ldots, S_n$. These blocks are moving blocks.

Let suppose that μ is a parameter of interest of the joint distribution of X_t, $t \in Z$ and $\hat{\theta}$ is a statistic. We calculate $Y_1=\hat{\theta}(S_1)$, $Y_2=\hat{\theta}(S_2),\ldots$, $Y_n=\hat{\theta}(S_n)$ and assume that $Y_1,Y_2,\ldots, Y_n$ are observations from a new time series Y_t. With these observations, in the same way, we can construct moving blocks compounded by b observations in the form $B_1=\{Y_1,\ldots,Y_b\}$, $B_2=\{Y_2,\ldots,Y_{b+1}\},\ldots,B_{n-b+1}=\{Y_{n-b+1},\ldots,Y_n\}$. We call them re-blocks.

Let $\overline{Y}_N = \frac{1}{n}\sum_{i=1}^{n} Y_i$ be the mean of the observations $Y_1,Y_2,\ldots,Y_n$ of the time series Y_t and $\overline{Y}_{-j}$ the mean of these observations in the case when we have deleted the observations of the j-is block. We denote $r=n-b+1$. Then we defined pseudovalues $J_j = \frac{1}{b}\left(n\overline{Y} - (n-b)\overline{Y}_{-j}\right)$, $j=1,2,\ldots,r$.

The jackknife estimation for the variance of $\sqrt{n}\,\overline{Y}_N$ is in the following form

$$\hat{v}_{JACK} = \frac{b}{r}\sum_{j=1}^{n}\left(J_j - \overline{Y}\right)^2$$

3. THE VALIDITY OF THE PROPOSED JACKKNIFE ESTIMATION

Let we do the following assumptions:

A1) $\{X_t, t \in Z\}$ is strictly stationary and α-mixing time series.

A2) $E|Y_1|^{2p+\delta} < c$, where p integer, p>2, $0 < \delta \le 2$ and c>0.

A3) $EY_1 < \mu + o\left(\frac{1}{\sqrt{n}}\right)$, where μ is a parameter of the joint distribution of X_t, $t \in Z$.

A4) $\sqrt{n}\left(\overline{Y} - E\overline{Y}\right) \xrightarrow{d} N\left(0, \sigma_\infty^2\right)$, $0 < \sigma_\infty^2 < \infty$.

Before we proof the theorem that shows the validity of the mentioned jackknife estimation we see the following lemma:

Lemma 1 *If time series* X_t *are* α-*mixing with mixing coefficient* $\alpha_X(k)$ *then* Y_t *are also* α-*mixing with mixing coefficient* $\alpha_Y(k) \le \alpha_X(k+s-1)$.

Proof. The distribution of $(Y_t,\dots,Y_k)$ depends on the distribution of $(X_t,\dots,X_{t+k+s-1})$, that not depended on t, because the time series X_t is strictly stationary.□

Lemma 2 *If* $\sum_{k=1}^{\infty} k^{p-1}\left(\alpha_X(k)\right)^{\delta/2p+\delta} < \infty$, *then* $\sum_{k=1}^{\infty} \alpha_X(k)^{p-2/p} < \infty$, *for* p *integer*, p>2 *and* $0 < \delta \le 2$.

Proof. For every $\varepsilon > 0$, exist such n_0, that for $n \ge n_0$, we have $k^{p-1}\left(\alpha_X(k)\right)^{\delta/2p+\delta} < \varepsilon$ or $\left(\alpha_X(k)\right)^{\delta/2p+\delta} < \frac{\varepsilon}{k^{p-1}}$. We can see that $\alpha_X(k)^{p-2/p} < \frac{\varepsilon_1}{k^{(p-1)(2p+\delta)(p-2)/p\delta}} < \frac{\varepsilon_1}{k^2}$. Now the proof is complete.□

To show the validity of the above variance jackknife estimation we prove the following theorem:

Theorem. *If* b=o(n) *and* $\sum_{k=1}^{\infty} k^{p-1}\left(\alpha_X(k)\right)^{\delta/2p+\delta} < \infty$, *for* p *integer*, p>2 *and* $0 < \delta \le 2$, *then* $\hat{v}_{JACK}\left(\sqrt{n}\overline{Y}_N\right) \xrightarrow{p} \sigma_\infty^2$.

Proof. We can see that $J_j = \frac{1}{b}\sum_{j=i}^{i+b-1} Y_j$, i=1,2,…,r. Let we denote $Z_i = \frac{1}{\sqrt{b}}\sum_{j=i}^{i+b-1} Y_j$, i=1,2,…,r. Then

$$\hat{v}_{JACK}\left(\sqrt{n}\overline{Y}_N\right) = \frac{b}{r}\sum_{i=1}^{r}\left(\frac{1}{b}\sum_{j=1}^{i+b-1} Y_j - \overline{Y}_N\right)^2 =$$

$$= \frac{1}{r}\sum_{i=1}^{r}\left(Z_j - \sqrt{b}\,\overline{Y}_N\right)^2 = \frac{1}{r}\sum_{i=1}^{r}\left(Z_i - EZ_i - \sqrt{b}\left(\overline{Y} - \frac{1}{\sqrt{b}}EZ_i\right)\right)^2 = A_N - 2C_N + B_N$$

where

$A_N = \frac{1}{r}\sum_{i=1}^{r}(Z_i - EZ_i)^2$, $\quad C_N = \frac{1}{r}\sum_{j=1}^{n}\sqrt{b}\left(\overline{Y} - \frac{1}{\sqrt{b}}EZ_i\right)(Z_i - EZ_i)$ and

$B_N = \frac{1}{r}\sum_{i=1}^{r} b\left(\overline{Y}_N - \frac{1}{\sqrt{b}}EZ_i\right)^2$.

Let we analyze alternately these terms:

1) $\mathrm{Var}A_N = \frac{1}{r}\mathrm{Var}(Z_1 - EZ_1)^2 + \frac{2}{r^2}\sum_{i=1}^{r-1}(r-i)\mathrm{cov}(Z_1 - EZ_1)(Z_{i+1} - EZ_{i+1})$

Combining some moment inequalities (Roussas and Ioanidis (1987)) and (Yokoyama (1980), Roussas (1988) and Yoshihara (1978)) and the assumption A1 we have

$$\mathrm{Var}A_N \le \frac{10}{r}\left(E|Z_1 - EZ_1|^{2p}\right)^{2/p}\left(\frac{1}{2}\right)^{(p-2)/p} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)\left(E|Z_1 - EZ_1|^{2p}\right)^{2/p}(\alpha_Y(i))^{(p-2)/p}$$

or

$$\mathrm{var}A_N \le \frac{10}{r}\left(K\left(E|Y_1|^{2p+\delta}\right)^{2p/2p+\delta}\right)^{2/p} +$$

$$+\frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)\left(K\left(E|Y_1|^{2p+\delta}\right)^{2p/2p+\delta}\right)^{2/p}(\alpha_Y(i))^{(p-2)/p} =$$

$$= O\left(\frac{1}{r} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)(\alpha_Y(i))^{(p-2)/p}\right)$$

Based on Lemma 1 and 2 we have $\mathrm{Var}A_N \xrightarrow{p} 0$. From the other hand, based on Chebyshev theorem, we have

$$EA_N = \mathrm{var}Z_1 = \mathrm{var}\left(\left(\frac{1}{\sqrt{b}}\right)\sum_{j=i}^{i+b-1}Y_j\right) = \mathrm{var}\left(\sqrt{b}\left(\frac{1}{b}\sum_{j=i}^{i+b-1}Y_j\right)\right) \xrightarrow{p} \sigma_\infty^2$$

2) Let we consider the term C_N.

$$C_N = \frac{1}{r}\sum_{j=1}^{n}\sqrt{b}\left(\overline{Y}_N - \frac{1}{\sqrt{b}}EZ_i\right)(Z_i - EZ_i) = \sqrt{b}\left(\overline{Y}_N - \frac{1}{\sqrt{b}}EZ_i\right)\frac{1}{r}\sum_{j=1}^{n}(Z_i - EZ_i)$$

So, in the same way followed above for the term A_N, we show that $\mathrm{var}\frac{1}{r}\sum_{j=1}^{n}(Z_i - EZ_i) \xrightarrow{p} 0$ and $E\frac{1}{r}\sum_{j=1}^{n}(Z_i - EZ_i) \xrightarrow{p} 0$.

$$\mathrm{var}\left(\frac{1}{r}\sum_{j=1}^{n}(Z_i - EZ_i)\right) = \frac{1}{r}\mathrm{var}(Z_i - EZ_i) + \frac{2}{r^2}\sum_{i=1}^{r-1}(r-i)\mathrm{cov}(Z_1 - EZ_1, Z_i - EZ_i) \le$$

$$\le \frac{10}{r}\left(E|Z_1 - EZ_1|^{p}\right)^{2/p}\left(\frac{1}{2}\right)^{(p-2)/p} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)\left(E|Z_1 - EZ_1|^{p}\right)^{2/p}(\alpha_Y(i))^{(p-2)/p}$$

$$\leq \frac{10}{r}\left(K\left(E\left|Y_1\right|^{2p+\delta}\right)^{2p/p+\delta}\right)\left(\frac{1}{2}\right)^{1/p} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)\left(K\left(E\left|Y_1\right|^{p+\delta}\right)^{2p/p+\delta}\right)(\alpha_Y(i))^{(p-2)/p} =$$

$$= O\left(\frac{1}{r} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)(\alpha_Y(i))^{(p-2)/p}\right)$$

Based on Lemma 1 and 2 we have $\operatorname{var}\frac{1}{r}\sum_{j=1}^{n}(Z_i - EZ_i)\xrightarrow{p}0$. From the other hand $E\frac{1}{r}\sum_{j=1}^{n}(Z_i - EZ_i)\xrightarrow{p}0$. Based on Chebyshev theorem we have $C_N \xrightarrow{p} 0$. To finish the theorem proof we consider the third term D_N.

3) $$D_N = \frac{1}{r}\sum_{i=1}^{r} b\left(\overline{Y}_N - \frac{1}{\sqrt{b}}EZ_i\right)^2 = \frac{1}{r}\sum_{i=1}^{r} b\left(\overline{Y}_N - \mu\right)^2 +$$

$$+ b\left(\mu - \frac{1}{\sqrt{b}}EZ_i\right)^2 + 2b\left(\overline{Y}_N - \mu\right)\left(\mu - \frac{1}{\sqrt{b}}EZ_i\right)$$

Let analyze separately the three terms:

a) Since b=o(n) and r=n-b+1 we have b=o(r). From A2 and A3 we take

$$\sqrt{b}\left(\overline{Y}_N - \mu\right) = \sqrt{b}\left(\overline{Y}_N - E\overline{Y}_N\right) + \sqrt{b}E\left(\overline{Y}_N - \mu\right) = \frac{\sqrt{b}}{\sqrt{r}}\sqrt{r}\left(\overline{Y}_N - E\overline{Y}_N\right) + \sqrt{b}o\left(r^{-1/2}\right)\xrightarrow{p}0$$

$$b\left(\overline{Y}_N - \mu\right)^2 = \frac{b}{N}\left(\sqrt{N}\left(\overline{Y}_N - \mu\right)\right)^2 \xrightarrow{p} 0$$

b) $$b\left(\overline{Y}_N - \mu\right)\left(\mu - \frac{1}{\sqrt{b}}EZ_i\right) = b\left(\overline{Y}_N - \mu\right)\left(\mu - EY_1\right) = o_p\left(br^{-1}\right)\xrightarrow{p}0$$

c) $$b\left(\mu - \frac{1}{\sqrt{b}}EZ_i\right)^2 = o\left(br^{-1}\right)\xrightarrow{p}0$$

So $D_N \xrightarrow{p} 0$. The proof is complete.□

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Free Ternary Semicommutative Groupoids

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Abstract. *Free objects in the variety V of ternary groupoids defined by the identity $[xyz] \approx [zyx]$, are described. A proper characterization of the class of V-free objects by means of the class of V-injective groupoids is obtained as well.*

Key words: *ternary semicommutative groupoid, free/injective ternary semicommutative groupoid.*

1. INTRODUCTION AND PRELIMINARIES

An algebra with one ternary operation is called a ***ternary groupoid*** and is denoted by $(G,[\,])$. A ternary groupoid $(G,[\,])$ is ***semicommutative*** if $[xyz]=[zyx]$ for all $x,y,z \in G$ ([1]). The class of ternary semicommutative groupoids is a variety defined by the identity $[xyz] \approx [zyx]$ and it is denoted by V. We will give a description of free objects in this variety using the absolutely free ternary groupoid $\mathbf{F}_X = (F,[\,])$.

For the sake of completeness we will briefly present the construction of the absolutely free ternary groupoid $\mathbf{F}_X = (F,[\,])$ that is a special case of the construction of an absolutely free (n,m)-groupoid over a nonempty set X, that does not contain the symbol $[\,]$, for $n=3$ and $m=1$ ([4]). Let $F_0, F_1, ..., F_p, ...$ be a sequence of disjoint sets such that

$$F_0 = X \ , \ F_1 = \{(x_1,x_2,x_3) \mid x_1,x_2,x_3 \in F_0\}, \ldots,$$

$$F_{p+1} = \{(t_1,t_2,t_3) \mid t_1,t_2,t_3 \in F_0 \cup F_1 \cup \ldots \cup F_p \wedge (\exists i \in \{1,2,3\}) \ t_i \in F_p\}.$$

If $t \in F_p$, then we say that t has a ***hierarchy*** p and denote it by $\chi(t)$.

Put $F = \bigcup \{F_p : p \geq 0\}$ and define a ternary operation $[\,]: F^3 \to F$ by:

$$t_1,t_2,t_3 \in F \ \Rightarrow \ [t_1,t_2,t_3] = (t_1,t_2,t_3) .$$

Then $\mathbf{F}_X = (F, [\,])$ is a ternary groupoid. The set X is the generating set for $\mathbf{F}_X$ and $\mathbf{F}_X$ has the universal mapping property for the class of ternary groupoids over X. Therefore, $\mathbf{F}_X = (F, [\,])$ is free over X in the class of ternary groupoids. The groupoid $\mathbf{F}_X$ is *injective*, since the operation $[\,]$ is an injective mapping, i.e. $[a_1, a_2, a_3] = [b_1, b_2, b_3] \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$. The set X is the set of prime elements in $\mathbf{F}_X$. An element $u \in F$ is said to be ***prime*** in $\mathbf{F}_X$ if and only if $u \neq [t_1, t_2, t_3]$ for all $t_1, t_2, t_3 \in F$. These two properties characterize all absolutely free ternary groupoids ([2], [3]).

Proposition 1.1. *A ternary groupoid* $(H, [\,])$ *is absolutely free if and only if* $(H, [\,])$ *is injective and the set of primes in* $(H, [\,])$ *is nonempty and generates* $(H, [\,])$.

We refer to this proposition as Bruck Theorem for the class of all ternary groupoids.

Define the ***length*** $|t|$, the *set* $P(t)$ *of* ***subterms*** and the set $var(t)$ of variables, for every $t \in F$, inductively in the following way:

$$t \in X \Rightarrow |t| = 1, \quad t = [t_1 t_2 t_3] \Rightarrow |t| = |t_1| + |t_2| + |t_3|,$$
$$t \in X \Rightarrow P(t) = \{t\}, \quad t = [t_1 t_2 t_3] \Rightarrow P(t) = \{t\} \cup P(t_1) \cup P(t_2) \cup P(t_3),$$
$$t \in X \Rightarrow var(t) = \{t\}, \quad t = [t_1 t_2 t_3] \Rightarrow var(t) = var(t_1) \cup var(t_2) \cup var(t_3).$$

2. A CONSTRUCTION OF CANONICAL OBJECTS IN V

A ternary semicommutative groupoid $\mathbf{R} = (R, [\,]^*)$ is said to be ***canonical*** if the following conditions are satisfied:

(C0) $X \subseteq R \subseteq F$ and $t \in R \Rightarrow P(t) \subseteq R$;

(C1) $[t_1 t_2 t_3] \in R \Rightarrow [t_1 t_2 t_3]^* = [t_1 t_2 t_3]$;

(C2) $\mathbf{R}$ is a V-free groupoid over X.

Define an ordering in F, denoted by $\leq$, in the following way. Let X be a linearly ordered set and $t, u \in F$. If $t, u \in X$, then $t \leq u$ in $F \Leftrightarrow t \leq u$ in X; if $\chi(t) < \chi(u)$, then $t \leq u$; if $\chi(t) = \chi(u) \geq 1$ and $t \neq u$, where $t = [t_1 t_2 t_3]$, $u = [u_1 u_2 u_3]$, then lexicographical ordering is used, with elements earlier in the lexicographical ordering coming first, i.e.

$$t < u \Leftrightarrow [t_1 < u_1 \vee (t_1 = u_1 \wedge t_2 < u_2)] \vee (t_1 = u_1 \wedge t_2 = u_2 \wedge t_3 < u_3)].$$

By induction on hierarchy one can show that F is a linearly ordered set.

Define a set R by:

$$R = X \cup \{ t \in F \setminus X \mid [t_1 t_2 t_3] \in P(t) \Rightarrow t_1 \leq t_3 \}.$$

Proposition 2.1. a) $X \subset R \subset F$; $t \in R \Rightarrow P(t) \subseteq R$.

b) $t_1, t_2, t_3 \in R \Rightarrow \left([t_1 t_2 t_3] \notin R \Leftrightarrow t_3 < t_1 \right)$.

c) $t_1, t_2, t_3 \in F \Rightarrow \left([t_1 t_2 t_3] \in R \Leftrightarrow t_1, t_2, t_3 \in R \wedge t_1 \leq t_3 \right)$.

Define a ternary operation $[\,]^*$ in R by:

$$t_1, t_2, t_3 \in R \Rightarrow [t_1 t_2 t_3]^* = \begin{cases} [t_1 t_2 t_3], & \text{if } t_1 \leq t_3 \\ [t_3 t_2 t_1], & \text{if } t_3 < t_1. \end{cases}$$

By a direct verification one can show that $(R, [\,]^*)$ is a ternary semicommutative groupoid, the set of prime elements in $(R, [\,]^*)$ coincides with X and generates $(R, [\,]^*)$. Also, $(R, [\,]^*)$ has the universal mapping property for V over X. Namely, if $\mathbf{G} = (G, [\,]') \in V$ and $\lambda : X \to G$ is a mapping, then there is a homomorphism $\psi : \mathbf{R} \to \mathbf{G}$ such that $\psi|_X = \lambda$. Let $\varphi : \mathbf{F}_X \to \mathbf{G}$ be the homomorphism that extends λ and let $\psi = \varphi|_R$. It suffices to show that $\varphi([t_1 t_2 t_3]^*) = [\varphi(t_1)\varphi(t_2)\varphi(t_3)]'$ for any $t_1, t_2, t_3 \in R$. If $[t_1 t_2 t_3] \in R$, then $[t_1 t_2 t_3]^* = [t_1 t_2 t_3]$, and so, $\varphi([t_1 t_2 t_3]^*) = \varphi([t_1 t_2 t_3]) =$ $= [\varphi(t_1)\varphi(t_2)\varphi(t_3)]'$. If $[t_1 t_2 t_3] \notin R$, then $[t_1 t_2 t_3]^* = [t_3 t_2 t_1]$, and so,

$$\varphi([t_1 t_2 t_3]^*) = \varphi([t_3 t_2 t_1]) = [\varphi(t_3)\varphi(t_2)\varphi(t_1)]' = [\mathbf{G} \in V] = [\varphi(t_1)\varphi(t_2)\varphi(t_3)]'.$$

Thus, we have proved the following

Theorem 2.1. *The ternary groupoid $(R, [\,]^*)$ is V-free over X and has a canonical form.*

3. A CHARACTERISATION OF FREE TERNARY SEMICOMMUTATIVE GROUPOIDS

We investigate some properties concerning the nonprime elements in $(R, [\,]^*)$ that are essential for introducing the notion of V-injectivity, and especially in which case the equality $[t_1 t_2 t_3]^* = [u_1 u_2 u_3]^*$ is fulfilled. By considering all the cases, one obtains:

Proposition 3.1. *Let t be a nonprime element in $(R,[\,]^*)$ and let (t_1,t_2,t_3) be a triple of divisors of t in $(R,[\,]^*)$. Then (u_1,u_2,u_3) is a triple of divisors of t if and only if $u_2=t_2$ and $\{u_1,u_3\}=\{t_1,t_3\}$.*

Therefore, $t=[t_1t_2t_3]\in R$ has one triple of divisors (t_1,t_2,t_3) if $t_1=t_3$ and
two triples of divisors (t_1,t_2,t_3) and (t_3,t_2,t_1) if $t_1\neq t_3$. The triple (t_1,t_2,t_3) of divisors of t in $(R,[\,]^*)$ coincides with the triple (t_1,t_2,t_3) in $(F,[\,])$. This is called the *canonical triple of divisors* in $(R,[\,]^*)$.

By Prop.3.1. the equalities $[t_1t_2u]^*=[t_1t_2v]^*$ and $[ut_2t_3]^*=[vt_2t_3]^*$ are equivalent with $\{t_1,u\}=\{t_1,v\}$ and $\{u,t_3\}=\{v,t_3\}$, respectively, which implies that $u=v$. Also, $[t_1ut_3]^*=[t_1vt_3]^*$ implies that $u=v$. Therefore:

Proposition 3.2. *The ternary groupoid $(R,[\,]^*)$ is left, right and middle cancellative.*

We use Prop.3.1. to introduce the notion of V-injective ternary groupoid.

Definition 3.1. A ternary groupoid $\mathbf{H}=(H,[\,])$ is said to be V*-injective* if and only if $\mathbf{H}\in V$ and $[a_1a_2a_3]=[b_1b_2b_3]\Rightarrow a_2=b_2\wedge\{a_1,a_3\}=\{b_1,b_3\}$, for all $a_i,b_i\in H$, $i=1,2,3$.

By Prop.3.1 it follows that the V-canonical ternary groupoid $(R,[\,]^*)$ is V-injective. Every V-free ternary groupoid over X is isomorphic with $(R,[\,]^*)$ and thus, the following proposition holds.

Proposition 3.3. *Every V-free ternary groupoid over X is V-injective.*

We will use the following lemma in the proof of Bruck Theorem for the variety V.

Lemma 3.1. *Let $\mathbf{H}=(H,[\,])$ be V-injective ternary groupoid such that the set P of primes in $\mathbf{H}$ is nonempty and generates $\mathbf{H}$. If $C_0=P$, $C_1=[C_0C_0C_0]$, …, $C_{k+1}=\{a\in H\setminus P:$ (d_1,d_2,d_3) is a triple of divisors of $a\Rightarrow$*

$$\Rightarrow\{d_1,d_2,d_3\}\subset C_0\cup\ldots\cup C_k\wedge\{d_1,d_2,d_3\}\cap C_k\neq\varnothing\},$$

then $H=\bigcup\{C_k:k\geq 0\}$, where $C_k\neq\varnothing$ for every $k\geq 0$, and $C_i\cap C_j=\varnothing$ for $i\neq j$.

Proof. If $k=1$, then $a \in C_2$ if and only if $a=[d_1 d_2 d_3]$ for some d_1, d_2, $d_3 \in C_0 \cup C_1$ and at least one $d_i \in C_1$. Then $[d_1 d_2 d_3]$ belongs to some of the sets $[C_0 C_0 C_1]$, $[C_0 C_1 C_0]$, $[C_1 C_0 C_0]$, $[C_0 C_1 C_1]$, $[C_1 C_0 C_1]$, $[C_1 C_1 C_0]$, $[C_1 C_1 C_1]$. Therefore, $a \in C_2$ if and only if a is in the union of these sets, i.e. C_2 equals to that union. Inductively, the set C_{k+1}, for any $k \geq 1$, is equal to the union of the sets $[C_p C_q C_r]$, where $p,q,r \in \{0,1,\ldots,k\}$ and at least one of the sets C_p, C_q, C_r is the set C_k. Hence, $C_i \cap C_{k+1} = \varnothing$. Therefore, $C_i \cap C_j = \varnothing$, for $i \neq j$. Since P generates $\mathbf{H}$, one can put $H = \bigcup_{k \geq 0} P_k$, where $P_0 = P$, $P_{k+1} = P_k \cup [P_k P_k P_k]$. By induction on k one can show that $P_k = C_0 \cup C_1 \cup \ldots \cup C_k$. Hence, $H = \bigcup \{C_k : k \geq 0\}$, where $C_k \neq \varnothing$, for any $k \geq 0$, and $C_i \cap C_j = \varnothing$ for $i \neq j$.

Theorem 3.1. (Bruck theorem for V) *A semicommutative ternary groupoid $\mathbf{H}$ is V-free if and only if $\mathbf{H}$ is V-injective and the set P of prime elements in $\mathbf{H}$ is nonempty and generates $\mathbf{H}$.*

Proof. The “if part” follows directly from Prop.3.3 and the fact that $\mathbf{H} \cong \mathbf{R}$. For the “only if part” we use Lemma 3.1.: $H = \bigcup \{C_k : k \geq 0\}$. It remains to show that $\mathbf{H}$ has the universal mapping property for V over P. Let $\mathbf{G} = (G, [\,]')$ be a semicommutative ternary groupoid and $\lambda : P \to G$ be a mapping. Define a sequence of mappings $\varphi_k : C_k \to G$ for $k \geq 0$, inductively by: $\varphi_0 = \lambda$ and let $\varphi_i : C_i \to G$ be defined for every $i \leq k$. If $a \in C_{k+1}$ and (d_1, d_2, d_3) is a triple of divisors of a, where $d_1 \in C_p$, $d_2 \in C_q$, $d_3 \in C_r$, then $p,q,r \leq k$. Putting $\varphi_{k+1}([d_1 d_2 d_3]) = [\varphi_k(d_1)\varphi_k(d_2)\varphi_k(d_3)]'$, we obtain that $\varphi = \cup\{\varphi_i : i \geq 0\}$ is a mapping from H into G that is a homomorphism. If $a = [d_1 d_2 d_3] \in H$, then (d_1, d_2, d_3) is a triple of divisors of a in $\mathbf{H}$ and $\varphi(a) = \varphi([d_1 d_2 d_3]) = [\varphi_k(d_1)\varphi_k(d_2)\varphi_k(d_3)]' = [\varphi(d_1)\varphi(d_2)\varphi(d_3)]'$. Since $\mathbf{H}$ is V-injective it follows that (d_3, d_2, d_1) is a triple of divisors of a as well and that $\varphi([d_3 d_2 d_1]) = [\varphi(d_3)\varphi(d_2)\varphi(d_1)]' = [G \in \mathcal{K}_{sc}] = [\varphi(d_1)\varphi(d_2)\varphi(d_3)]' =$

$= \varphi([d_1 d_2 d_3])$. Thus, φ is a well defined mapping and a homomorphism. Hence, $\mathbf{H}$ is V-free over P.

The class of V-injective ternary groupoids is wider than the class of V-free groupoids, as the following example shows.

Let X be a countable set and let $(R, [\,]^*)$ be the V-canonical ternary groupoid over X. Define a set $H = \{t \in R : |P(t)| = 1\} \subset R$, a set $D \subset H^3$ by $D = \{(t_1, t_2, t_3) \in H^3 : P(t_i) \neq P(t_j) \text{ for a pair } i, j \in \{1,2,3\}\}$ and a relation θ in D by $(t_1, t_2, t_3)\,\theta\,(u_1, u_2, u_3) \Leftrightarrow t_2 = u_2 \wedge \{t_1, t_3\} = \{u_1, u_3\}$. It is easy to show that θ is an equivalence relation in D. The equivalence class $(t_1, t_2, t_3)^\theta = \{(t_1, t_2, t_3), (t_3, t_2, t_1)\}$, when $t_1 \neq t_3$ and $(t_1, t_2, t_3)^\theta = \{(t_1, t_2, t_3)\}$, when $t_1 = t_3$. The set of θ-equivalent classes is denoted by D^θ. It has the same cardinality as the set X and thus, there is an injection $\psi : D^\theta \to X$. Define an operation $[\,]'$ on H by: for $t_1, t_2, t_3 \in H$,

$$[t_1 t_2 t_3]' = \begin{cases} [t_1 t_2 t_3]^*, \text{ if } P(t_1) = P(t_2) = P(t_3) \\ \psi((t_1, t_2, t_3)^\theta), \text{ if } P(t_i) \neq P(t_j), \text{ for some pair } i, j \in \{1,2,3\}. \end{cases}$$

By a direct verification one can show that $\mathbf{H} = (H, [\,]')$ is a ternary groupoid that satisfies the conditions of Def.3.1. In the cases when the mapping ψ is an bijection, the set P of prime elements in $\mathbf{H}$ is empty. By Thm.3.1, $\mathbf{H} = (H, [\,]')$ is not V-free. Hence:

Proposition 3.4. *The class of V-free ternary groupoids is a proper subclass of the class of V-injective groupoids.*

This shows that Thm.3.1. gives a proper characterization of the class of free semicommutative ternary groupoids by means of the class of V-injective groupoids.

The obtained results can be easily generalized for semicommutative n-ary groupoids, i.e. n-ary groupoids defined by the identity $[x_1 x_2 \ldots x_{n-1} x_n] \approx [x_n x_2 \ldots x_{n-1} x_1]$.

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Intrinsic Shape of Unstable Attractors

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***Abstract**: We shall investigate the shape of unstable attractors using the intrinsic approach to shape. Natural analogy will be provided to lead us from the shape properties of stable to unstable attractors.*
***Keywords**: intrinsic shape, proximate net, attractor*

1. INTRODUCTION

Exploring the structures of sets which captures the long term behavior of the system with topological tools is of great importance. Stable attractors are greatly investigated and shape properties in ANR-spaces are elegantly described in [1] and [2] for example. On the other hand when unstable attractors are in question a shortage of results is felt. One good paper which enriches us with some answers and deals with these sets with shape and homological tools is [3]. Following the intrinsic approach to shape for compact spaces given in [4] we raise the question about the shape properties of these sets using the naturally constructed proximate net from the flow in the system.

A flow in a metric space (X,d) is one parameter family of homeomorphisms $\{\varphi_t \mid t\in\mathbb{R}\}$, $\varphi_t : X \to X$ such that they satisfy the following two conditions: $\varphi_0 = 1_X$, $\varphi_t \circ \varphi_s = \varphi_{t+s}, \forall s,t \in \mathbb{R}$. A set $M \subseteq X$ is invariant under the flow if $\phi(M,t) \subseteq M$, $\forall t \in \mathbb{R}$. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$ or $\mathbb{R}^-$ we obtain the corresponding notions of positive and negative invariant set. A compact invariant set M is stable if every neighborhood U of M contains a positively invariant neighborhood V of M. Positive limit set of a given subset $M \subseteq X$ is: $\omega(M) = \{x \mid \exists x_n \in M, \exists t_n \to +\infty, \ \varphi(x_n,t_n) \to x\}$.

Analogous we define negative limit set $\alpha(M)$. If invariant set M admits neighborhood U such that the maximal invariant subset of U is actually M then we call M isolated invariant and U isolating neighborhood.

We are ready to introduce the concept of attractor via the notion of basin of attraction: $A(M) = \{x \in M \mid \varnothing \neq \omega(x) \subseteq M\}$.

We say that a compact invariant set M is an attractor if its basin of attraction is a neighborhood of M. If the attractor is stable set then we call it stable attractor. Unstable attractor is attractor which is not stable.

2. INTRINSIC SHAPE IN COMPACT METRIC SPACES

There are several approaches in building the shape category. Papers [4] and [5] are good references for the intrinsic approach to shape. We shall follow the construction given in [4]. Let X, Y be compact metric spaces.

Definition. A function is $f : X \to Y$ is $\varepsilon -$continuous, if for every $x \in X$ there is a neighborhood of x whose image lies in the $\varepsilon -$open ball of the image of x.

Definition. Two functions $f, g : X \to Y$ are $\varepsilon -$homotopic if there exists $\varepsilon -$continuous function $F : X \times I \to Y$ such that for every point $x \in X$, $F(x,0) = f(x)$, $F(x,1) = g(x)$.

The relation of $\varepsilon -$homotopy is an equivalence relation in the set of $\varepsilon -$continuous functions.

Definition. A proximate net from X to Y is a sequence of not necessary continuous functions $f_n : X \to Y$ such that for every $\varepsilon > 0$, there is an index n_0 such that f_n is $\varepsilon -$homotopic to f_{n+1}, for every $n \geq n_0$.

We denote proximate nets by $(f_n) : X \to Y$ or just with (f_n).

Definition. Two proximate nets (f_n) and (g_n) are homotopic if for every $\varepsilon > 0$, f_n is $\varepsilon -$homotopic to g_n for almost every n.

We denote homotopy classes by $[(f_n)]$. We choose as objects compact metric spaces and as morphisms homotopy classes of proximate nets. It only remains to define composition of classes of proximate nets to construct the shape category.

Direct composition of two proximate nets need not be a proximate net as the following example shows:

Example: Let $X = Y = Z = [0,1]$. We define proximate net $(f_n) : X \to Y$ and proximate net $(g_n) : Y \to Z$ by:

$$f_n(x) = \begin{cases} \frac{1}{n}, x \in [\frac{1}{n}, 1] \\ 0, x \in [0, \frac{1}{n}) \end{cases}, \quad g_n(x) = \begin{cases} nx, x \in [0, \frac{1}{n}] \\ 1, \ x \in [\frac{1}{n}, 1] \end{cases}$$

The composition $h_n(x)=(g_n\circ f_n)(x)=\begin{cases}1, x\in[\frac{1}{n},1]\\ 0, x\in[0,\frac{1}{n})\end{cases}$ however does not form a proximate net $(h_n): X\to Z$. The solution for this problem is choosing an appropriate subsequence as shown in [4] by which a unique proximate net is obtained up to homotopy. So for given proximate nets $(f_n),(g_n)$ we define composition of classes of homotopy by:

$$[(g_n)]\circ[(f_n)]=[(g_n\circ f_{k_n})]$$

3. INTRINSIC SHAPE IN FLOWS

We will need the construction of shape morphism in a given flow. Let us recall that if a stable attractor M is given then a sequence of real numbers (t_n) exists such that: $\varphi_{t_n}(x)=\varphi(x,t_n)\in T(M,\frac{1}{n}),\forall x\in A(M)$, where the notation $T(M,\frac{1}{n})$ stands for open ball at M. A question is imposed: Is it possible to obtain such a sequence of real numbers (t_n) for unstable attractors? The answer is of course negative, but we can try something else. Instead of sequence we can adjoin a function $t_n(x)$ such that in a neighborhood U of M we have that:

(1) $$\varphi_{t_n(x)+t}(x)=\varphi(x,t_n(x)+t)\in T(M,\frac{1}{n}),\forall t>0$$, or

(2) $$\varphi_{t_n(x)-t}(x)=\varphi(x,t_n(x)-t)\in T(M,\frac{1}{n}),\forall t>0$$

Before making this idea precise let us define a map for a given compact invariant set M in the following way: Let $x\in X$, $t\in\mathbb{R}$ are arbitrary. We go with the flow from x until the point $\varphi_t(x)=\varphi(x,t)$. Now we measure the distance $d(\varphi(x,t),M)$ which by compactness of M is achieved in some point $m_x^t\in M$. Of course this point may not be unique, but never the less we pick any such point. So we can define a map $m: X\times\mathbb{R}\to M$ by setting $m(x,t)=m_x^t$.

Let (α_n) is monotonically decreasing zero sequence i.e. $\lim_{n\to\infty}\alpha_n = 0$. For a compact isolated invariant set M let us suppose that there exists compact isolating neighborhood U such that for the local base $U_n = T(M,\alpha_n)$ for M, there exists a functional sequence (τ_n), $\tau_n : U \to \mathbb{R}$ from continuous functions such that the following holds:

- $\varphi(x,\tau_n(x)) \in S(M,\alpha_n), \forall x \in U \setminus U_n, \forall n \in \mathbb{N}$, where $S(M,\alpha_n)$ denote the sphere centered at M.
- For arbitrary $x \in U$ we have that:

$$(\varphi(x,\tau_n(x)+t) \in U_n, \forall t > 0, \forall n \in \mathbb{N}) \vee (\varphi(x,\tau_n(x)-t) \in U_n, \forall t > 0, \forall n \in \mathbb{N})$$

Theorem1. The inclusion map $i : M \to U$ is shape equivalence.

Proof: We define proximate nets as follows: using the inclusion map $i : M \to U$ and setting for maps $i_n = i$, for arbitrary $n \in \mathbb{N}$ we consider $(i_n) : M \to U$ which is obviously proximate net. Now we define the maps $f_n : U \to M$ by $f_n(x) = m(x,\tau_n(x))$.

Lemma1. The sequence $(f_n) : U \to M$ is a proximate net.

Proof: Let $\varepsilon > 0$ be given and define a map $H_n : U \times \mathrm{I} \to M$ with:

$$H_n(x,t) = m(x,(1-t)\tau_n(x) + t\tau_{n+1}(x))$$

Let us note that $H_n(x,0) = f_n(x)$, $H_n(x,1) = f_{n+1}(x)$, so we only need to prove that for sufficiently large n, H_n is $\varepsilon-$continuous. We shall use the following claim which is easy to prove:

Claim1. Let $f : X \to Y$ be a given map between compact metric spaces, $\varepsilon > 0$ is arbitrary small number and $p \in X$. If for arbitrary (p_n) such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} f(p_n)$ exists, the following holds:

$$\lim_{n\to\infty} f(p_n) \in T(f(p),\varepsilon)$$

then the map f is $\varepsilon-$continuous at p.

We'll discuss the continuity of H according to the point's location. For $(p,t) \in \operatorname{int} M \times \mathrm{I}$ the continuity is clear. Let's consider points $(p,t) \in \partial M \times \mathrm{I}$. We choose arbitrary sequence (p_k,t_k) which converge to the point (p,t). If $p_k \in M$ then we surely have $\lim_{k\to\infty} H(p_k,t_k) = H(p,t)$. If $p_k \notin M$ then we shall consider two cases:

(3) $$\varphi(p_k,\tau_n(p_k)+t)\in U_n, \forall t>0, \forall n\in\mathbb{N}, \forall k\in\mathbb{N};$$

(4) $$\varphi(p_k,\tau_n(p_k)-t)\in U_n, \forall t>0, \forall n\in\mathbb{N}, \forall k\in\mathbb{N};$$

Let us assume that the first case (3) is valid. If $\tau_{n+1}(p_k)>\tau_n(p_k)$ then from the inequality: $\tau_n(p_k)(1-t_k)+\tau_{n+1}(p_k)t_k>\tau_n(p_k)$ we have that:

$$d(\varphi(p_k,\tau_n(p_k)(1-t_k)+\tau_{n+1}(p_k)t_k), m(p_k,\tau_n(p_k)(1-t_k)+\tau_{n+1}(p_k)t_k))<\alpha_n$$

Now using the triangle inequality we have that:

$$d(H_n(p_k,t_k),H_n(p,t))\le d(H_n(p_k,t_k),\varphi(p_k,\tau_n(p_k)(1-t_k)+\tau_{n+1}(p_k)t_k))+$$
$$d(\varphi(p_k,\tau_n(p_k)(1-t_k)+\tau_{n+1}(p_k)t_k),H_n(p,t))<\alpha_n+\beta_k<\alpha_n+\beta_k+\frac{1}{n}.$$

for some zero sequence β_k. If $\tau_{n+1}(p_k)\le\tau_n(p_k)$ similarly we obtain that: $d(H_n(p_k,t_k),H_n(p,t))<\alpha_{n+1}+\beta_k+\frac{1}{n}$. The second case (4) is similar. So we can conclude that in all cases we surely have:

$$d(H_n(p_k,t_k),H_n(p,t))<\alpha_n+\alpha_{n+1}+\beta_k+\frac{1}{n},$$

for some zero sequence β_k. Pushing things to the limit we have that: $\lim\limits_{k\to\infty}H_n(p_k,t_k)\in T(H_n(p,t),\alpha_n+\alpha_{n+1}+\frac{1}{n})$. But this means that we have $\alpha_n+\alpha_{n+1}+\frac{1}{n}$-continuity in this points. With similar discussion we obtain that for points (p,t) such that $p\notin M$ we have $2\alpha_n+2\alpha_{n+1}+\frac{1}{n}$-continuity which finishes the proof of the lemma.

Now it remains to prove the following relations:

(5) $$[(f_n)]\circ[(i_n)]=[(1_M)],\ [(i_n)]\circ[(f_n)]=[(1_U)]$$

The first composition is obvious. Namely, from the definition we have $[(f_n)]\circ[(i_n)]=[(f_n\circ i_{k_n})]=[(f_n\circ i)]$ and having in mind that the map $f_n\circ i:M\to M$ is actually $(f_n\circ i)(x)=f_n(i(x))=f_n(x)=\varphi(x,\tau_n(x))$, the homotopy $H_n:M\times\mathrm{I}\to M$ defined by $H_n(x,t)=\varphi(x,(1-t)\tau_n(x))$ settles the job.

It remains to prove the second composition from (5). From the definition we have that $[(i_n)]\circ[(f_n)]=[(i_n\circ f_{k_n})]$. The map $i_n\circ f_{k_n}:U\to U$ is given by $i_n(f_{k_n}(x))=f_{k_n}(x)=m(x,\tau_{k_n}(x))$. We'll define homotopy $H_n:U\times \mathrm{I}\to U$ by:

$$H_n(x,t)=\begin{cases}\varphi(x,\tau_{k_n}(x)(1-t)), & x\in M\vee(x\notin M\wedge t\neq 0)\\ m(x,\tau_{k_n}(x)), & x\notin M\wedge t=0\end{cases}$$

We shall prove that for arbitrary small $\varepsilon>0$ the map H_n is $\varepsilon-$continuous for sufficiently large n. For points $(x,t),t\neq 0$ the continuity of H_n is clear. We'll discuss only the points (x,t) such that $x\in\partial M,t=0$. The other cases are similar. Let's choose arbitrary sequence (x_p,t_p) such that it converges to the selected point (x,t). If $x_p\in M$ then it is clear that $\lim\limits_{p\to\infty}H_n(x_p,t_p)=H_n(x,t)$. If the points $x_p\notin M$ and $t_p\neq 0$ then similarly we have $\lim\limits_{p\to\infty}H_n(x_p,t_p)=H_n(x,t)$. The remaining possibility is $x_p\notin M,\ t_p=0$. For sufficiently large n the following holds:

$d(\varphi(x_p,\tau_{k_n}(x_p)),m(x_p,\tau_{k_n}(x_p)))\leq\alpha_{k_n}<\dfrac{\varepsilon}{4}$. Also for sufficiently large p the following holds: $d(\varphi(x,\tau_{k_n}(x)),\varphi(x_p,\tau_{k_n}(x_p)))<\dfrac{\varepsilon}{4}$ which gives us, using the triangle inequality, the following estimate:

$$d(H_n(x_p,0),H_n(x,0))\leq d(H_n(x_p,0),\varphi(x_p,\tau_{k_n}(x_p)))+$$

$$+d(\varphi(x_p,\tau_{k_n}(x_p)),H_n(x,0))<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}$$

Pushing things to the limit we have $\lim\limits_{p\to\infty}H_n(x_p,0)\in T(H_n(x,0),\varepsilon)$. The proof is complete.

Now using a result from [3] which proves existence of such functions $\tau_n(x)$ if the explosion points (defined in [3]) of an unstable attractor M are internal, we can state the following corollary:

Corollary1. Every isolated unstable attractor M with internal explosions only, admits neighbourhood U with the same shape, i.e. $Sh(U)=Sh(M)$.

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New derivative-free nonmonotone line search methods for unconstrained minimization

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Abstract: *Two new derivative-free nonmonotone line search methods for unconstrained optimization are proposed and analyzed. Convergence is established under standard conditions. Numerical results show good performance of the proposed methods.*
Keywords: *unconstrained optimization, nonmonotone line search, derivative-free methods*
AMS subject classification: 65K05, 90C56

1 Introduction

Let us consider the problem of unconstrained optimization:

$$\min_{x\in\mathbb{R}^n} f(x), \qquad (1)$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is bounded from below and has continuous partial derivatives that are not available.

A line search method for solving the problem (1), generates a sequence of iterates $\{x_k\}\in\mathbb{R}^n$ by the iterative formula $x_{k+1}=x_k+\alpha_k d_k$, where d_k is a search direction at x_k and $\alpha_k>0$ is a step size which is usually chosen to minimize the objective function f along the direction d_k. For more details about line search methods see [6]. The majority of line search methods require decrease in f at each iteration, which means that the corresponding sequence of function values $\{f(x_k)\}=\{f_k\}$ monotonically decreases. This may sometimes result in a slow convergence. On the other hand, the nonmonotone line search methods tend to converge faster and avoid converging to local minima, see [3, 4, 8, 9]. All of the above mentioned nonmonotone methods require the gradient of the objective function f and are thus unsuitable for the problem (1), or for problems where the objective function f is not smooth. In such situations, derivative-free methods based only on values of the objective function are more appropriate, see [6].

Diniz-Ehrhardt et al. [2] have proposed a derivative-free line search strategy which combines and extends the ideas form [3] and [4]. For given sequences $\{\eta_k\}$ and $\{\beta_k\}$, $k = 0,1,2,\ldots$ such that:

$$\eta_k > 0, \sum_{k=0}^{\infty}\eta_k = \eta < \infty \text{ and } \beta_k > 0, \lim_{k\in K}\beta_k = 0 \Rightarrow \lim_{k\in K}\nabla f(x_k) = 0, \qquad (2)$$

where ∇f is the gradient of f and $K \subseteq \mathbb{N}$ is an infinite set of indices, and a direction d_k, the step size $\alpha_k > 0$ is determined using the following nonmonotone line search rule:

$$f(x_k + \alpha_k d_k) \le \bar{f}_k + \eta_k - \alpha_k^2 \beta_k, \qquad (3)$$

with $\bar{f}_k = \max\{f(x_{k-j}) \mid 0 \le j \le m(k) - 1\}$, where $m(k) = \min\{k+1, M\}$ for $k = 0,1,2,\ldots$ and some $M \in \mathbb{N}$. We will refer to this method as M - *method*. In [2], the convergence with probability 1 is established for the M - method, when the search directions are randomly chosen and independent, bounded and descent with a fixed probability $p > 0$.

In this paper, we propose new derivative-free nonmonotone line search methods based on the line search rule (3) where the term $\bar{f}_k$ is chosen differently. We will establish the same convergence result as in [2], under same conditions. Numerical results show that the proposed new methods are competitive with the one from [2]. The rest of the paper is organized as follows: the new methods are formulated in section 2, while their convergence is established in section 3. The numerical results are presented in section 4.

2 New nonmonotone line search rules

Assume that the sequences $\{\eta_k\}$ and $\{\beta_k\}$ satisfy the conditions (2). Let $\{r_k\}$, $r_k \in [0,1]$ for all $k = 0,1,\ldots$. The first new line search rule has the form of (3), where $\bar{f}_k$ is chosen to be $\bar{f}_k = C_k$, where $\{Q_k\}$ and $\{C_k\}$ are recursive sequences defined by:

$$Q_{k+1} = r_k Q_k + 1, \qquad C_{k+1} = \frac{r_k Q_k (C_k + \eta_k) + f_{k+1}}{Q_{k+1}}, \qquad (4)$$

with $Q_0 = 1$, $C_0 = f_0$. We will refer to it as C_k - *line search rule* and to the corresponding method as C_k *-method*. The idea for a sequence $\{C_k\}$ comes from Zhang and Hager [9], and is used by many authors, e.g. [1].

The second line search rule also has the form of (3), where $\bar{f}_k$ is defined by $\bar{f}_k = \max\{f_k, \sum_{r=0}^{m(k)-1}\lambda_{k_r} f_{k-r}\}$, where $m(k) = \min\{k+1, M\}$ for

$k = 0,1,2,\ldots$ and some $M \in \mathbb{N}$, and λ_{k_r} are such that $\lambda_{k_r} \geq \lambda,\ r = 0,1,\ldots,m(k)-1$ and $\sum_{r=0}^{m(k)-1} \lambda_{k_r} = 1,$ for all $k = 0,1,\ldots$ and some $\lambda \in (0,1]$. We will refer to it as λ *-line search rule* and to the corresponding method as λ *-method*. The idea of a convex combination of last M functional values has origin in Ulbrich [7], but it has not been given much attention until in [8].

Now we state the model algorithm for C_k - and λ -method.

Algorithm 1 (Model algorithm (C_k-method / λ -method))

Choose $x_0 \in \mathbb{R}^n$ and sequences $\{\eta_k\}$ and $\{\beta_k\}$ that satisfy the conditions (2).

Compute the search direction d_k such that $\|d_k\| \leq \Delta$.

Compute the term $\bar{f}_k$ (according to C_k -method / λ -method).

Choose $0 < \alpha_k \leq 1$ such that $f(x_k + \alpha_k d_k) \leq \bar{f}_k + \eta_k - \alpha_k^2 \beta_k$.

Set $x_{k+1} = x_k + \alpha_k d_k$, $k = k+1$ and go to Step 1.

Let us note that $\eta_k > 0$ ensures that the line search rule in Step 3 is satisfied for a sufficiently small step size α_k, so the Algorithm 1 is well defined.

3 Convergence results

First we will prove some useful properties of the C_k - and λ -method. Let $\{x_k\}$ be an iterative sequence generated by Algorithm 1 using the C_k -method. Then $f_k \leq C_k \leq C_{k-1} + \eta_{k-1}$ for all $k \in \mathbb{N}$.

Proof. For all $k \in \mathbb{N}$, by Step 3 in Algorithm 1 we have $f_k \leq C_{k-1} + \eta_{k-1} - \alpha_{k-1}^2 \beta_{k-1} \leq C_{k-1} + \eta_{k-1}$. Then, the proof proceeds as the proof of Lemma 2.2 in [1].

Let $\{x_k\}$ be an iterative sequence generated by Algorithm 1 using the λ -method. Then:

$$f_k \leq f_0 + \sum_{j=0}^{k-1} \eta_j - \lambda \sum_{j=0}^{k-2} \alpha_j^2 \beta_j - \alpha_{k-1}^2 \beta_{k-1} \leq f_0 + \sum_{j=0}^{k-1} \eta_j - \lambda \sum_{j=0}^{k-1} \alpha_j^2 \beta_j .$$

Proof. The proof is by induction. For $k=1$, since $\lambda \le 1$, and Step 3 from Algorithm 1 we have $f_1 \le f_0 + \eta_0 - \alpha_0^2 \beta_0 \le f_0 + \eta_0 - \lambda \alpha_0^2 \beta_0$. Assume that the assumption is true for all j, $1 \le j \le k$. We have two cases:

Case 1. $\max\{f_k, \sum_{r=0}^{m(k)-1} \lambda_{k_r} f_{k-r}\} = f_k$

Case 2. $\max\{f_k, \sum_{r=0}^{m(k)-1} \lambda_{k_r} f_{k-r}\} = \sum_{r=0}^{m(k)-1} \lambda_{k_r} f_{k-r}$

The proof can be completed by following the technique of the proof of Lemma 1 in [8], with additional consideration of the conditions (2).

Let us note that Lemma 3 and Lemma 3 imply that a sequence $\{x_k\}$ generated by Algorithm 1 is such that $x_k \in \{x \in \mathbb{R}^n \mid f(x) \le f_0 + \eta\}$ for all k. The next lemma gives another useful property shared by the C_k- and λ-method. Let $\{x_k\}$ be an iterative sequences generated by Algorithm 1. Then, there exists an infinite subset of indices $K \subseteq \mathbb{N}$ such that $\lim_{k \in K} \alpha_k^2 \beta_k = 0$.

Proof. If C_k-method is used we have:

$$C_{k+1} = \frac{r_k Q_k (C_k + \eta_k) + f_{k+1}}{Q_{k+1}} \le \frac{r_k Q_k (C_k + \eta_k) + C_k + \eta_k - \alpha_k^2 \beta_k}{Q_{k+1}} = C_k + \eta_k - \frac{\alpha_k^2 \beta_k}{Q_{k+1}}.$$

Now, summing up the first $k+1$ inequalities we have $C_{k+1} \le C_0 + \sum_{j=0}^{k} \eta_j - \sum_{j=0}^{k} \dfrac{\alpha_j^2 \beta_j}{Q_{j+1}}$. From Lemma 3, since f is bounded from below with a constant $K > 0$, for all k we have:

$$\sum_{j=0}^{k} \frac{\alpha_j^2 \beta_j}{Q_{j+1}} \le C_0 + \sum_{j=0}^{k} \eta_j - C_{k+1} \le f_0 + \eta - f_{k+1} \le f_0 + \eta - K,$$

which implies that $\sum_{j=0}^{\infty} \dfrac{\alpha_j^2 \beta_j}{Q_{j+1}} < +\infty$. Since $Q_{j+1} \le j+2$, we have that $\liminf_{j \to \infty} \alpha_j^2 \beta_j = 0$.

If λ-method is used, by Lemma 3 we have:

$$f_k \le f_0 + \sum_{j=0}^{k-1} \eta_j - \lambda \sum_{j=0}^{k-1} \alpha_j^2 \beta_j \le f_0 + \eta - \lambda \sum_{j=0}^{k-1} \alpha_j^2 \beta_j.$$

Now, since f is bounded from below with a constant $K > 0$ and $\lambda > 0$, for all k we have:

$$\sum_{j=0}^{k-1} \alpha_j^2 \beta_j \le \frac{f_0 + \eta - f_{k+1}}{\lambda} \le \frac{f_0 + \eta - K}{\lambda},$$

which implies that $\sum_{j=0}^{\infty}\alpha_j^2\beta_j < +\infty$ and $\liminf_{j\to\infty}\alpha_j^2\beta_j = 0$, which completes the proof.

Let us note that, if $0 < r_k < 1$ for all k in C_k-method, then the stronger result $\lim_{j\to\infty}\alpha_j^2\beta_j = 0$ can be proven. Now we state one of the main theorems for the proposed derivative-free nonmonotone line search methods. Since $\bar{f}_k \geq f_k$ is valid for both C_k- and λ-method, the proof follows from Lemma 3, mimicing the proof of Theorem 1 in [2]. Let $\{x_k\}$ be an iterative sequence generated by Algorithm 1. Assume that (x^*, d) is a limit point of the subsequence $\{(x_k, d_k)\}_{k\in K}$ where $K \subseteq \mathbb{N}$ is an infinite subset of indices. Then $\nabla f(x^*)^T d \geq 0$. Using Lemma 3, under additional assumptions for the search directions d_k, such as: $\|d_k\| \in [\Delta_{\min}, \Delta_{\max}]$ and $\nabla f(x_k)^T d_k \leq -\theta \|\nabla f(x_k)\| \; \|d_k\|$ for infinitely many k, where $0 < \Delta_{\min} < \Delta_{\max} < \infty$ and $0 < \theta < 1$, it can be proven that Algorithm 1 finds stationary points up to any arbitrary precision (see Corollary 1 in [2]). In order to have truly derivative-free methods, we can relax the above additional conditions in such a way that the search directions are randomly chosen, independent, bounded and descent with a fixed probability $p > 0$. This way, convergence with probability 1 to a stationary point can be established (see Theorem 2 in [2]).

4 Numerical results

In this section we present some of the results from testing M-, C_k-, λ- and the ``monotone'' line search method (the last one by using $M = 1$ in (3)). All test problems are from Moré et al. [5] and all of them have optimal function value $f^* = 0$. The sequences in (2) have been chosen such that $\beta_k \equiv 1$ and $\eta_k = f_0/k^{1.1}$. Search direction $d_k = -\hat{g}_k/\sigma_k$ is used, where $\hat{g}_k$ is a discrete gradient approximation of $\nabla f(x_k)$, and σ_k is the spectral coefficient (see [2]). The rest of parameters for M-, C_k-, λ-method are $M = 5$, $r_k = 0.85$ for all k and $\lambda_{k_r} = 1/m(k)$ for all r. The results are reported in Table 1, where *It* denotes the number of iterations, *Evalf* denotes number of function evaluations, f is the function value at the last iterate, *normg* is the norm of the gradient approximation at the last iterate. All methods tested on the first problem (MGH26) stopped after the maximum of 5000 iterations was reached, all methods except C_k-method tested on the second (MGH22) and third (MGH21) problem stopped after the maximum of 500000 function evaluations was exceeded, and the C_k -method tested on the second and third problem stopped when $|f_k| \leq 10^{-9}$. Additional numerical results are also available at http://www.institutzamatematika.com/index.php/Irena_Stojkovska_Curriculum_Vitae. All results show good performance of the proposed new derivative-free nonmonotone line search methods compared with the M -method. There are many problems for which nonmonotone methods converge faster to the solution than the ``monotone'' one, which is as expected.

5 Conclusions

In this paper two new derivative-free nonmonotone line search methods were proposed and their convergence established under the same assumptions as in [2]. Numerical results show good performance of the proposed methods. The fact that nonmonotone methods are able to avoid local minima, leads to the idea to test these nonmonotone methods on noisy functions.

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Prb	n	It	Evalf	f	normg
		``Monotone'' line search / M -method, [2]			
MGH26	10	5000 / 5000	422555 / 422885	$2.80E-05$ / $2.80E-05$	$2.36E-05$ / $7.40E-06$
MGH22	100	2428 / 2427	500182 / 500048	$4.21E-06$ / $1.22E-04$	$3.81E-03$ / $2.19E-01$
MGH21	100	2424 / 2425	500073 / 500137	$1.02E-05$ / $5.52E-07$	$2.88E-03$ / $6.65E-04$
		C_k -method / λ -method			
MGH26	10	5000 / 5000	382384 / 342569	$2.80E-05$ / $2.80E-05$	$6.41E-05$ / $6.21E-06$
MGH22	100	533 / 2429	109040 / 500165	$4.25E-11$ / $1.22E-04$	$1.43E-06$ / $5.92E-03$
MGH21	100	1907 / 2444	390414 / 500193	$9.97E-10$ / $2.26E-08$	$3.03E-05$ / $1.47E-04$

Table 1: Results for tested line search methods with spectral gradient direction

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Structural Description Of Generalized $(m+k,m)$-Rectangular Bands

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Abstract: *A definition and a structural description of the generalized* $(m+k,m)$ *-rectangular bands are given.*

Keywords: $(m+k,m)$ *-rectangular band, generalized* $(m+k,m)$ *-rectangular band.*

1. INTRODUCTION

We introduce some notations which will be used further on:

1) The elements of Q^s, where Q^s denotes the $s-$th Cartesian power of Q, will be denoted by x_1^s.
2) The symbol x_i^j will denote the sequence $x_i x_{i+1} ... x_j$ when $i \le j$, and the empty sequence when $i > j$.
3) If $x_1 = x_2 = ... = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.
4) The set $\{1,2,...,s\}$ will be denoted by $\mathbf{N_s}$.

Let $Q \neq \varnothing$ and n, m be positive integers. We say that a map $[\ ]$ from Q^n into Q^m, is an (n,m)-operation. A pair $(Q;[\ ])$ where $[\ ]$ is an (n,m)-operation is called an (n,m)-groupoid. Every (n,m)-operation on Q induces a sequence $[\ ]_1, [\ ]_2, ..., [\ ]_m$ of n-ary operations on the set Q, such that

$$((\forall i \in \mathbf{N_m})\ \ [x_1^n]_i = y_i) \Leftrightarrow [x_1^n] = y_1^m .$$

Let $m \ge 2, k \ge 1$. An $(m+k,m)$-groupoid $(Q;[\ ])$ is called an $(m+k,m)$-semigroup if for each $i \in \{0,1,2,...,k\}$

$$[x_1^i [x_{i+1}^{i+m+k}] x_{i+m+k+1}^{m+2k}] = [[x_1^{m+k}] x_{m+k+1}^{m+2k}].$$

Let $(L;[\ ])$ be an $(m+k,m)$-groupoid, where $[\ ]$ is an $(m+k,m)$-operation defined by $[x_1^{m+k}] = x_1^m$. Then $(A;[\ ])$ is an $(m+k,m)$-semigroup and it is called a left-zero $(m+k,m)$-semigroup. Dually, a right-zero $(m+k,m)$-semigroup $(R;[\ ])$ is defined by the operation $[x_1^{m+k}]=x_{k+1}^{m+k}$.

The pair $(L\times R;[\ ])$, where $[\ ]$ is an $(m+k,m)$-operation on $L\times R$ defined by

$$[x_1^{m+k}]=y_1^m \Leftrightarrow (x_i=(a_i,b_i), y_j=(a_j,b_{j+k}), i\in \mathbf{N_{m+k}},\ j\in \mathbf{N_m})$$

is an $(m+k,m)$-semigroup and it is a direct product of a left-zero and a right-zero $(m+k,m)$-semigroups L and R. Such an $(m+k,m)$-semigroup is called an $(m+k,m)$-rectangular band.

The next proposition gives a characterization of $(m+k,m)$-rectangular bands as $(m+k,m)$-semigroups in which three identities are satisfied ([3], [4]).

Proposition 1.1. Let $\mathbf{Q}=(Q;[\ ])$ be an $(m+k,m)$-semigroup. $\mathbf{Q}$ is an $(m+k,m)$-rectangular band if and only if for all $i,j\in \mathbf{N_m}$, the following conditions are satisfied in $\mathbf{Q}$:

(RB I) $\left[x_1^{m+k}\right]_i=\left[y_1^{j-1}x_i y_{j+1}^{j+k-1}x_{i+k}y_{j+k+1}^{m+k}\right]_j$,

(RB II) $\left[x_1^{m+2k}\right]_i=\left[x_1^i x_{i+k+1}^{m+2k}\right]_i$,

(RB III) $\left[\overset{m+k}{x}\right]=\overset{m}{x}$.

2. GENERALIZED $(m+k,m)$-RECTANGULAR BANDS

A generalization of $(m+k,m)$-rectangular band is made by omitting the identity (RB III).

Definition 2.1. An $(m+k,m)$-semigroup $(Q;[\ ])$ in which identities (RB I) and (RB II) are satisfied is called a generalized $(m+k,m)$-rectangular band.

Proposition 2.2. Let $\mathbf{Q}=(Q;[\ ])$ be a generalized $(m+k,m)$-rectangular band, that is not an $(m+k,m)$-rectangular band. Then, there are a proper $(m+k,m)$-subsemigroup $(R;[\ ])$ of $\mathbf{Q}$ and a map $\psi:O\to R$, where $O=Q\setminus R$, such that:

1) $(R;[\ ])$ is an $(m+k,m)$-rectangular band; and

2) For each $x_1^{m+k}\in Q^{m+k}$,

$$\left[x_1^{m+k}\right]=[\varphi(x_1)\varphi(x_2)...\varphi(x_{m+k})],$$

where $\varphi:Q\to R$ is defined by

$$\varphi(x)=\begin{cases}\psi(x), & x\in O\\ x, & x\in R\end{cases}.$$

Proof. Let $x_1^{m+k}\in Q^{m+k}$, $i,j\in \mathbf{N_m}$. Then:

$$\left[\overset{m+k}{[x_1^{m+k}]_i}\right]_j \overset{\text{RB I}}{=} \left[[x_1^{m+k}]_i\,\overset{k-1}{a}\,[x_1^{m+k}]_i\,\overset{m-1}{a}\right]_1$$

$$\overset{\text{RB I}}{=} \left[[x_1^{m+k}]_i\,\overset{k-1}{a}\left[x_i\,\overset{k-1}{a}\,x_{i+k}\,\overset{m-1}{a}\right]_1\overset{m-1}{a}\right]_1$$

$$\overset{\text{RB I}}{=} \left[[x_1^{m+k}]_i\,\overset{k-1}{a}\left[x_i\,\overset{k-1}{a}\,x_{i+k}\,\overset{m-1}{a}\right]_1\left[x_i\,\overset{k-1}{a}\,x_{i+k}\,\overset{m-1}{a}\right]_2\ldots\left[x_i\,\overset{k-1}{a}\,x_{i+k}\,\overset{m-1}{a}\right]_m\right]_1$$

$$=\left[[x_1^{m+k}]_i\,\overset{k-1}{a}\,x_i\,\overset{k-1}{a}\,x_{i+k}\,\overset{m-1}{a}\right]_1 \overset{\text{RB II}}{=} \left[[x_1^{m+k}]_i\,\overset{k-1}{a}\,x_{i+k}\,\overset{m-1}{a}\right]_1$$

$$\overset{\text{RB I}}{=} \left[\overset{m-1}{a}\,[x_1^{m+k}]_i\,\overset{k-1}{a}\,x_{i+k}\right]_m \overset{\text{RB I}}{=} \left[\overset{m-1}{a}\left[\overset{m-1}{a}\,x_i\,\overset{k-1}{a}\,x_{i+k}\right]_m\overset{k-1}{a}\,x_{i+k}\right]_m$$

$$\overset{\text{RB I}}{=} \left[\left[\overset{m-1}{a}\,x_i\,\overset{k-1}{a}\,x_{i+k}\right]_1\left[\overset{m-1}{a}\,x_i\,\overset{k-1}{a}\,x_{i+k}\right]_2\ldots\left[\overset{m-1}{a}\,x_i\,\overset{k-1}{a}\,x_{i+k}\right]_m\overset{k-1}{a}\,x_{i+k}\right]_m$$

$$=\left[\overset{m-1}{a}\,x_i\,\overset{k-1}{a}\,x_{i+k}\,\overset{k-1}{a}\,x_{i+k}\right]_m \overset{\text{RB II}}{=} \left[\overset{m-1}{a}\,x_i\,\overset{k-1}{a}\,x_{i+k}\right]_m \overset{\text{RB I}}{=} [x_1^{m+k}]_i\,.$$

So, for each $x_1^{m+k}\in Q^{m+k}$ and each $i\in \mathbf{N_m}$, $\left[\overset{m+k}{[x_1^{m+k}]_i}\right]=\overset{m}{[x_1^{m+k}]_i}$. This implies that the subset R of Q defined by $R=\left\{x\,\middle|\,x\in Q,\left[\overset{m+k}{x}\right]=\overset{m}{x}\right\}$ is an $(m+k,m)$-subsemigroup of $(Q;[\ ])$, and the identities (RB I) and (RB II) are satisfied in $(R;[\ ])$. Since $(Q;[\ ])$ is not an $(m+k,m)$-rectangular band, it follows that R is a proper subset of Q, i.e. the set $O=Q\setminus R$ is not the empty set. The definition of R implies that the identity (RB III) is satisfied in $(R;[\ ])$. All this, together with Proposition 1.1., implies that $(R;[\ ])$ is an $(m+k,m)$-rectangular band.

If $\varphi : Q \to Q$ is defined by $\varphi(x) = \left[\overset{m+k}{x}\right]_1$, the above discussion implies that $\varphi : Q \to R$, and that $\varphi|_R = 1_R$. Let $\psi : O \to R$ be the restriction of φ on O.

Next, for each $x_1^{m+k} \in Q^{m+k}$:

$$
\begin{aligned}
[\varphi(x_1)\varphi(x_2)\ldots\varphi(x_{m+k})]_i &\overset{\mathbf{RB\,I}}{=} \left[\varphi(x_i)\overset{k-1}{a}\varphi(x_{i+k})\overset{m-1}{a}\right]_1 = \left[\left[\overset{m+k}{x_i}\right]_1 \overset{k-1}{a}\left[\overset{m+k}{x_{i+k}}\right]_1 \overset{m-1}{a}\right]_1 \\
&\overset{\mathbf{RB\,I}}{=} \left[\left[\overset{m+k}{x_i}\right]_1 \overset{k-1}{a}\left[\overset{m+k}{x_{i+k}}\right]_1\left[\overset{m+k}{x_{i+k}}\right]_2 \ldots \left[\overset{m+k}{x_{i+k}}\right]_m\right]_1 = \left[\left[\overset{m+k}{x_i}\right]_1 \overset{k-1}{a}\,\overset{m+k}{x_{i+k}}\right]_1 \\
&\overset{\mathbf{RB\,II}}{=} \left[\left[\overset{m+k}{x_i}\right]_1 \overset{m+k-1}{x_{i+k}}\right]_1 \overset{\mathbf{RB\,I}}{=} \left[\left[\overset{m+k}{x_i}\right]_1 \overset{k-1}{a}\,x_{i+k}\overset{m-1}{a}\right]_1 \\
&\overset{\mathbf{RB\,I}}{=} \left[\overset{m-1}{a}\left[\overset{m+k}{x_i}\right]_1 \overset{k-1}{a}\,x_{i+k}\right]_m \overset{\mathbf{RB\,I}}{=} \left[\overset{m-1}{a}\left[\overset{m-1}{a}\,x_i\overset{k-1}{a}\,x_i\right]_m \overset{k-1}{a}\,x_{i+k}\right]_m \\
&\overset{\mathbf{RB\,I}}{=} \left[\left[\overset{m-1}{a}\,x_i\overset{k-1}{a}\,x_i\right]_1\left[\overset{m-1}{a}\,x_i\overset{k-1}{a}\,x_i\right]_2 \ldots \left[\overset{m-1}{a}\,x_i\overset{k-1}{a}\,x_i\right]_m \overset{k-1}{a}\,x_{i+k}\right]_m \\
&= \left[\overset{m-1}{a}\,x_i\overset{k-1}{a}\,x_i\overset{k-1}{a}\,x_{i+k}\right]_m \overset{\mathbf{RB\,II}}{=} \left[\overset{m-1}{a}\,x_i\overset{k-1}{a}\,x_{i+k}\right]_m \overset{\mathbf{RB\,I}}{=} \left[x_1^{m+k}\right]_i . \ \blacksquare
\end{aligned}
$$

Next we will give a construction of generalized $(m+k,m)$-rectangular bands, as a converse to Proposition 2.2.

Let O and R be two non empty disjoint sets, and let $(R;[\ ])$ be an $(m+k,m)$-rectangular band. Let $Q = R \cup O$ and $\psi : O \to R$ be an arbitrary map. We extend ψ to the map $\varphi : Q \to R$ defined by:

$$
\varphi(x) = \begin{cases} \psi(x), & x \in O \\ x, & x \in R \end{cases},
$$

and we define an $(m+k,m)$-operation on Q, by:

$$
\left[x_1^{m+k}\right] = [\varphi(x_1)\varphi(x_2)\ldots\varphi(x_{m+k})].
$$

Proposition 2.3. The $(m+k,m)$-groupoid $\mathbf{Q} = (Q;[\ ])$ defined above is a generalized $(m+k,m)$-rectangular band, that does not satisfy (RB III).

Proof. For $x_1^{m+2k} \in Q^{m+2k}$, the definition of the $(m+k,m)$-operation on Q implies:

$$[[x_1^{m+k}]x_{m+k+1}^{m+2k}]_i = [\varphi([x_1^{m+k}]_1)\varphi([x_1^{m+k}]_2)\ldots\varphi([x_1^{m+k}]_m)\varphi(x_{m+k+1})\varphi(x_{m+k+2})\ldots\varphi(x_{m+2k})]_i$$

$$= [\varphi([\varphi(x_1)\ldots\varphi(x_{m+k})]_1)\ldots\varphi([\varphi(x_1)\ldots\varphi(x_{m+k})]_m)\varphi(x_{m+k+1})\ldots\varphi(x_{m+2k})]_i .$$

Since $[\varphi(x_1)\ldots\varphi(x_{m+k})]_j \in R$, for each $j \in \mathbf{N_m}$, it follows that

$$\varphi([\varphi(x_1)\ldots\varphi(x_{m+k})]_j) = [\varphi(x_1)\ldots\varphi(x_{m+k})]_j .$$

Therefore,

$$[\varphi([\varphi(x_1)\ldots\varphi(x_{m+k})]_1)\ldots\varphi([\varphi(x_1)\ldots\varphi(x_{m+k})]_m)\varphi(x_{m+k+1})\ldots\varphi(x_{m+2k})]_i$$

$$= [[\varphi(x_1)\ldots\varphi(x_{m+k})]_1\ldots[\varphi(x_1)\ldots\varphi(x_{m+k})]_m\varphi(x_{m+k+1})\ldots\varphi(x_{m+2k})]_i .$$

The assumption that $(R;[\ ])$ is an $(m+k,m)$-semigroup implies that:

$$[[\varphi(x_1)\ldots\varphi(x_{m+k})]_1\ldots[\varphi(x_1)\ldots\varphi(x_{m+k})]_m\varphi(x_{m+k+1})\ldots\varphi(x_{m+2k})]_i$$

$$= [\varphi(x_1)\ldots\varphi(x_{m+k})\varphi(x_{m+k+1})\ldots\varphi(x_{m+2k})]_i$$

$$= [\varphi(x_1)\ldots\varphi(x_j)[\varphi(x_{j+1})\ldots\varphi(x_{j+m+k})]_1\ldots[\varphi(x_{j+1})\ldots\varphi(x_{j+m+k})]_m\varphi(x_{j+m+k+1})\ldots\varphi(x_{m+2k})]_i$$

$$= [\varphi(x_1)\ldots\varphi(x_j)\varphi([\varphi(x_{j+1})\ldots\varphi(x_{j+m+k})]_1)\ldots\varphi([\varphi(x_{j+1})\ldots\varphi(x_{j+m+k})]_m)\varphi(x_{j+m+k+1})\ldots\varphi(x_{m+2k})]_i$$

$$= \left[\varphi(x_1)\ldots\varphi(x_j)\varphi\left(\left[x_{j+1}^{j+m+k}\right]_1\right)\ldots\varphi\left(\left[x_{j+1}^{j+m+k}\right]_m\varphi(x_{j+m+k+1})\ldots\varphi(x_{m+2k})\right)\right]_i$$

$$= \left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i .$$

The above discussion shows that $[[x_1^{m+k}]x_{m+k+1}^{m+2k}]_i = [x_1^j[x_{j+1}^{j+m+k}]x_{j+m+k+1}^{m+2k}]_i$ for each $i \in \mathbf{N_m}$, $j \in \mathbf{N_k}$, i.e. that $(Q;[\ ])$ is an $(m+k,m)$-semigroup.

The identities (RB I) and (RB II) are satisfied in $(Q;[\ ])$, because:

$$[x_1^{m+k}]_i = [\varphi(x_1)\varphi(x_2)\ldots\varphi(x_{m+k})]_i$$

$$= [\varphi(y_1)\ldots\varphi(y_{j-1})\varphi(x_i)\varphi(y_{j+1})\ldots\varphi(y_{j+k-1})\varphi(x_{i+k})\ldots\varphi(y_{m+k})]_j$$

$$= [y_1^{j-1}x_i y_{j+1}^{j+k-1}x_{i+k}y_{j+k+1}^{m+2k}]_j$$

and

$$\left[x_1^{m+2k}\right]_i = [\varphi(x_1)\varphi(x_2)...\varphi(x_{m+2k})]_i = [\varphi(x_1)...\varphi(x_i)\varphi(x_{i+k+1})...\varphi(x_{m+2k})]_i$$

$$= \left[x_1^i x_{i+k+1}^{m+2k}\right]_i .$$

Hence, $(Q;[\ ])$ is a generalized $(m+k,m)$-rectangular band.

Since O is nonempty set, let $a \in O$ and let $\varphi(a) = b$. Since O and R are disjoint, $a \neq b$. Then, the definition of the $(m+k,m)$-operation on Q and the assumption that $(R;[\ ])$ is an $(m+k,m)$-rectangular band, imply that $\left[\overset{m+k}{a}\right] = \left[\overset{m+k}{\varphi(a)}\right] = \left[\overset{m+k}{b}\right] = \overset{m}{b} \neq \overset{m}{a}$. Hence, **Q** does not satisfy (RB III). ■

A generalized $(m+k,m)$-rectangular band **Q** $= (Q;[\ ])$ defined by the above construction, will be denoted by **Q** $= [R,O,\psi]$.

Propositions 2.2. and 2.3. imply the following theorem.

Theorem 2.4. An $(m+k,m)$-semigroup **Q** $= (Q;[\ ])$ is a generalized $(m+k,m)$-rectangular band that is not an $(m+k,m)$-rectangular band if and only if **Q** $= [R,O,\psi]$, for some $(m+k,m)$-rectangular band $(R;[\ ])$, nonempty set O disjoint from R, and a map $\psi : O \to R$. ■

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Bootstrap Confidence Intervals for the Fractional Difference Parameter in ARFIMA Model

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Abstract: *Many empirical time series have been found to exhibit behavior characterized by the presence of the long-range dependence, alias the long memory. The presence of long memory, initially observed in data analysis in hydrology, is established in diverse fields, for example in financial economics, networks traffic, psychology, cardiology, etc. One, but not the only, model commonly used to fit a time series with long memory is ARFIMA model. In this paper we consider a block bootstrap method, which uses blocks composed of cycles, to construct confidence intervals for the fractional differential parameter in ARFIMA models. A simulation Monte Carlo study is conducted to obtain the performance of these bootstrap confidence intervals.*
Keywords: *Time series, long memory, ARFIMA, fractional parameter, confidence interval, block bootstrap.*

1. INTRODUCTION

It has already been established that many empirical time series data exhibit behavior characterized by a stronger persistence of their correlations than for data generated by usual short run linear models, such as the vast class of ARMA (*Auto Regression Moving Average*) processes. Such data have occurred in the applied sciences such as hydrology, finance, networks traffic etc.The concept of the long memory or the long range dependence describes this property. Perhaps the most known and commonly used class of parametric models of long memory processes is the class of ARFIMA(p,d,q) (*Auto Regression Fractionally Integrated Moving Average*) models, proposed by Hosking ([9]). A crucial issue about model inferences is the point estimation or interval estimation of the fractional differential parameter d, since it indicates the intensity of the memory.

This paper presents a bootstrap technique applied to obtain confidence intervals for the fractional differential parameter in ARFIMA models. A simulated Monte Carlo study is conducted to estimate empirically the performance of these bootstrap confidence intervals. The outline of the paper is as follows. In Section 2 we specify definitions of long memory and

ARFIMA process. Then we describe two semi-parametric estimators for fractional parameter. In section 3 we consider a review of the bootstrap resampling techniques for time series and their application on constructing confidence intervals. In Section 4 we report a simulation study comparing the performance of proposed bootstrap intervals in respect to empirical coverage probability and interval length.

2. LONG MEMORY AND ARFIMA MODELS

2.1. The long memory notion

Long range dependence and long memory are synonymous notions that have recently become especially important. However there is not a unique theoretical definition of them. When definitions are given, they vary from author to author. Most of the definitions of long range dependence or long memory appearing in literature are based on the second order properties of a stochastic process. Such properties include asymptotic behavior of the autocovariance function or the spectral density function.

Let $\{X_t\}_{t\in Z}$ be a stationary time series with real values. It is intuitively expected that the autocorrelation function vanishes when the distance between the data becomes large. So we can suppose that the data are asymptotically independent and the autocorrelations are absolutely summable. It is the case in most of stationary time series included the vast class of ARMA models, referred as short memory processes. In contrast, long memory is generally defined by the fact that the autocorrelations are absolutely non-summable. However, there are alternative definitions not necessarily equivalent to one-another (for more details see [1]). Following Beran ([1]) let us give a definition of long memory defined in spectral domain.

Let $\{X_t\}_{t\in Z}$ be a stationary process with a spectral density function $f(\omega)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma(k)e^{ik\omega}$ (here $\gamma(\bullet)$ denotes the autocovariance function). Suppose that there exists a real number $\beta\in(0,1)$ and a constant $c_f>0$ not depended on ω such that

$$(1)\qquad \lim_{\omega\to 0}\frac{f(\omega)}{c_f\,|\omega|^{-\beta}}=1$$

Then $\{X_t\}_{t\in Z}$ is called a stationary process with long memory or long-range dependence or strong dependence.

2.2. ARFIMA(p,d,q) models.

It is often said in literature that the ARFIMA models can describe well a stationary series with long memory behavior. This model is a generalization of an ARIMA(p,d,q) model (see [2]) obtained by permitting the differential parameter, d to take any real value instead of being restricted to integer values. A stationary process $\{X_t\}_{t\in Z}$ is called ARFIMA (p,d,q) process if it is defined as follows:

(2) $\Phi(B)(1-B)^d X_t = \Theta(B)\varepsilon_t$

Here B is the backshift operator and $\{\varepsilon_t\}$ is a white noise with zero mean and finite variance σ_ε^2. If the polynomials $\Phi(\bullet)$ and $\Theta(\bullet)$ have their roots outside the unit circle and $-0.5 < d < 0.5$, then the process $\{X_t\}_{t\in Z}$ is stationary and invertible. When $0 < d < 0.5$ the process exhibits long memory in the sense of equation (1) with $\beta = 2d$. When $-0.5 < d < 0$ we say that the process has intermediate memory and, when $d = 0$ we have the ARMA(p,q) process (short memory). In the rest of the paper we will focus on the simplest ARFIMA model obtained when $p = q = 0$ and on the case when $0 < d < 0.5$. In this simple case equation (2) defines an ARFIMA(0,d,0) process also called a FI(d) (*Fractionally Integrated*) process with the parameter d.

2.3. Estimation of the fractional parameter.

There have been proposed several estimators for the fractional parameter in the literature. In our simulation study, we use two semiparametric estimators based on the regression equation constructed from the logarithm of the spectral density function (see [6], [12]). One of the most used estimators is proposed by Geweke and Porter-Hudak ([6]) and is usually noted as GPH estimator. This estimator is based on the log-periodogram regression $u_j = a + bv_j + e_j$, where

$u_j = \ln\left(I(\omega_j)\right)$, $a = \ln\left(f_{ARMA}(0)\right)$, $b = -d$, $v_j = \ln\left(4\sin^2\left(\omega_j/2\right)\right)$ and,

$e_j = \ln\left(\dfrac{I(\omega_j)}{f(\omega_j)}\right), j = 1,2,...,g$ are the random errors. Here $f_{ARMA}(\omega)$ denotes the spectral density function of the corresponding ARMA(p,q) process of (2). Function $I(\omega)=\frac{1}{2\pi}\left[R(0)+2\sum_{k=1}^{n-1}R(k)\cos(k\omega)\right]$ is the periodogram of the observed data $x_1,x_2,...,x_n$ and $\omega_j=\frac{2\pi j}{n}$, $j=1,2,...,g$ are the Fourier frequencies. Then, *d* can be estimated by $\hat{d}_{GPH}=-\hat{b}=-\frac{\sum_{j=1}^{g}\left(v_j-\overline{v}\right)u_j}{\sum_{j=1}^{g}\left(v_j-\overline{v}\right)^2}$, where $\overline{v}=\frac{1}{g}\sum_{j=1}^{g}v_j$. An important practical problem in implementation of the GPH estimator is the choice of the truncation parameter $g=g(n)$, which has an important influence on the bias and variance of the estimator.

Reisen ([12]) suggested using a smoothed periodogram instead of the sample periodogram obtaining the smoothed periodogram (SP) estimator. Both GPH and SP estimators are asymptotically unbiased and normally distributed.

3. THE BOOTSTRAP METHODS FOR TIME SERIES.

The bootstrap methodology, proposed originally by Efron [4] is an effective technique to present solutions when the parametric methods and statistical theory do not work. But, Efron’s bootstrap classical application in the context of dependent data, such as a time series, fails to work. In the last years many bootstrap methods for time series have been developed. The most common are the block bootstrap methods. They consider a set of consecutive observations to define blocks. Then, the bootstrap time series is constructed by resampling (with replacement) blocks and concatenating them still yielding the bootstrap time series. Different methods differ in the way as blocks are constructed. For a theoretical comparison of some methods see [11]. In this paper we consider blocks that are composed of one or more consecutive cycles. Subtracting the mean from each term of the series a cycle is defined as a pair of alternating positive and negative terms. We consider both the non-overlapping block bootstrap (NBB) and the moving block bootstrap (MBB). In the MBB case a cycle is considered as an inseparable observation. The number of cycles of a block is a tuning parameter that should be determined appropriately. Bootstrap methods for time series have been widely used to build confidence intervals (see [3], [5]), [7]). We consider the construction of different types of bootstrap intervals for the fractional parameter in the following simulation study.

4. SIMULATION STUDY

Firstly, we simulated 2000 Monte Carlo ARFIMA(0,d,0) series for different values of the parameter d and with different sample sizes, to calculate the average cycle length. From results, not shown in this paper, we observe that the cycle length depends strongly on the memory parameter and very slightly on the sample size. Then, we simulated ARFIMA(0,d,0) processes for values of parameter d=0.15,0.3,0.45 and with sample sizes n=100,128,300,500, removing 300 first terms from simulated series. For each simulated time series we applied both NBB and MBB methods to construct the bootstrap percentile (PC), percentile-t (Pt), bias corrected (BC) and bias corrected and accelerated (BCa) intervals (see [3], [7]) for the fractional parameter, based on the GPH and SP estimators. We used Q=1399 bootstrap replications (see [8]). Also we constructed the asymptotic (Asym) confidence intervals for the fractional parameter in each case. The nominal level of the intervals was chosen to be 0.95. We used $g(n)=n^{0.8}$ (see [10]) terms in the periodogram regression for GPH estimator. For SP estimator we used $g(n)=n^{0.5}$ regression terms and the values $m=n^{0.9}$ and $m=n^{0.7}$ in Parzen lag window (see [12]). We used different numbers of cycles per block. In each simulation case we calculated the average interval length (AIL) and the empirical coverage probability (ECP) in percentage for each type of intervals using 1000 Monte Carlo replications. R software and the "fracdiff" package were used. Table 1 contains some results based on GPH estimator with MBB method using a block length at order $O(n^{0.5})$.

Tab. 1: Bootstrap point and interval estimations based on GPH estimator with MBB.

	d	Point estimat.	PC Interval AIL[1] (ECP)[2]	Pt Interval AIL[1] (ECP)[2]	CB Interval AIL[1] (ECP)[2]	Asym Interval AIL[1] (ECP)[2]
n=100	0.15	0.11774	0.4883 (96.1)	0.4794 (72.7)	0.4346 (76.3)	0.4895 (94.3)
	0.30	0.26824	0.4914 (96.7)	0.4723 (73.4)	0.4374 (76.0)	0.4852 (93.9)
	0.45	0.41148	0.5002 (93.6)	0.4795 (70.5)	0.4453 (73.1)	0.4952 (92.3)
n=128	0.15	0.12119	0.4234 (96.6)	0.4164 (73.9)	0.3787 (75.0)	0.4340 (94.7)
	0.30	0.26986	0.4275 (95.4)	0.4148 (72.0)	0.3666 (74.0)	0.4324 (94.1)
	0.45	0.41618	0.4331 (94.4)	0.4178 (72.9)	0.3811 (75.6)	0.4139 (92.9)
n=300	0.15	0.13163	0.2842 (97.4)	0.2819 (78.4)	0.2590 (78.8)	0.2898 (95.4)
	0.30	0.28384	0.2849 (95.9)	0.2800 (75.3)	0.2545 (76.9)	0.2891 (93.8)
	0.45	0.43584	0.2873 (96.0)	0.2810 (74.7)	0.2560 (75.8)	0.2900 (94.4)
n=500	0.15	0.13588	0.2258 (96.3)	0.2248 (76.2)	0.2054 (77.2)	0.2296 (94.1)
	0.30	0.28816	0.2259 (96.9)	0.2225 (78.1)	0.2049 (78.7)	0.2282 (95.2)
	0.45	0.44551	0.2256 (97.2)	0.2211 (74.5)	0.2028 (75.8)	0.2284 (92.6)

[1] Average Interval Length. [2] Empirical Coverage Probability in percentage.

From all observed results we can summarize as following.

All the point estimations of *d* are downward biased, but this is more emphasized for SP estimations. Note that the BC and BCa intervals based on both GPH and SP estimators perform almost in the same way, so we refer only to BC intervals. Also, there are slight differences in the performance of the NBB and MBB methods. In almost all the cases the PC intervals based on the GPH estimator outperform the other intervals concerning both the ECP and the AIL. Their efficiency is more evident when they are compared with CB and Pt intervals based on GPH estimator or with all intervals based on the SP estimator. Also the PC intervals based on GPH estimator have in general a higher ECP than the corresponded asymptotic GPH intervals having approximately the same AIL as them. In most cases the ECP of PC intervals based on GPH estimator is higher than the nominal coverage probability. The performance of bootstrap confidence intervals tends to get better when the sample size increases.

The simulation results show that constructing bootstrap confidence intervals, based on the GPH estimator and applying the bootstrap with blocks composed of a few numbers of cycles, is a simple effective technique for the interval estimation of the fractional parameter of ARFIMA models. At least in our conducted models, these bootstrap intervals perform well even for small sample sizes of a time series with long memory.

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Description of (4,2)-equivalences

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Abstract: *Two characterizations of* (4,2) *– equivalences are given.*
Key words: *partitions,* (n,m)*-equivalences,* (4,2)*- equivalences.*

1. INTRODUCTION

Generalizing the notion of a partition on a set, J. Hartmanis introduced in [7] the notion of a partition of type n ($n\in \mathbf{N}$), for sets having at least n distinct elements.

Definition 1.1. ([7]) Let M be a set with more than $n-1$ elements. A family $\mathbf{P}_n(M)$ of subsets of M is called a ***partition of*** *M* ***of type*** *n* if:

(H1) For any $A\in \mathbf{P}_n(M)$ there is at least one finite subset B of M with n elements ($|B| = n$), such that $B\subseteq A$; and

(H2) For any $B\subseteq M$ with $|B| = n$ there is exactly one $A\in \mathbf{P}_n(M)$, such that $B\subseteq A$.

With this notion, the partitions of type 1 are the ordinary partitions of a set, and the partitions of type 2 are incidence geometries.

Analogously to the fact that ordinary partitions are equivalent to the equivalence relations, H. E. Picket in [8] described a family of (n+1)-ary equivalence relations for partitions of type n. For its definition we will use the following notations.

For a given set M and a $k\in \mathbf{N}$ we write $\mathbf{x}=a_1...a_k$ or just $\mathbf{x}=a_1^k$ instead $\mathbf{x}=(a_1,...,a_k)$ for elements $\mathbf{x}$ in the Cartesian power $\mathbf{x}\in M^k$. For $a\in M$, we write a^k instead of $(a,...,a)$. We denote by $S(k)$ the permutation group on k elements, and $card(a_1^k)=k$ means that all the elements a_j are distinct.

Definition 1.2. ([8]) A subset $\rho\subseteq M^{n+1}$ is called (n+1)***-ary equivalence relations*** on M if:

(P1) For each $a_1^n\in M^n$, $(a_1^n, a_1)\in \rho$;

(P2) For $a_1^{n+1}\in M^{n+1}$ and each permutation $\pi\in S(n+1)$, $a_1^{n+1}\in\rho$ implies $(a_{\pi(1)},...,a_{\pi(n+1)})\in\rho$; and

(P3) For each $a_1^{n+2} \in M^{n+2}$ with $card(a_2^{n+1}) = n$, $a_1^{n+1} \in \rho$ and $a_2^{n+2} \in \rho$ imply $(a_1^n, a_{n+2}) \in \rho$.

The condition (P2) allows us to consider an (n+1*)-ary equivalence relations* on M as a subset of the symmetric Cartesian power $M^{(n+1)}$ of M.

For $k \in \mathbf{N}$, the symmetric Cartesian power $M^{(k)}$ is the factor set $M^k / \approx$ where $\approx$ is the equivalence relation on M^k defined by: $a_1^k \approx b_1^k$ if there is a permutation $\pi \in S(k)$ such that $a_t = b_{\pi(t)}$, for any $1 \leq t \leq k$. We will write a_1^k instead $(a_1^k)^{\approx}$ for the elements of $M^{(k)}$.

Generalized equivalence relations are also considered, for example in [9], [10] and [12].

Connecting the notion of an n-metric, considered, by K. Menger [6], by S. Gähler [4], [5], and by J. Usan [11], incidence structures with n-metrics were examined in [2], and the notion of (n,m,ρ)-metrics together with the notion of (n,m)-equivalence relations were introduced in [1].

Definition 1.3. ([1]) For $n,m \in \mathbf{N}$, $m < n$, a subset $\rho \subseteq M^{(n)}$ is called (n,m)- ***equivalence relations*** (shortly (n,m)- ***equivalence***) on M if:

(Rn) $(\forall a \in M^n)\ a^n \in \rho$;

(Tnm) For each $\mathbf{x} \in M^{(n)}$, $\mathbf{b} \in M^{(m)}$, if $\mathbf{ub} \in \rho$ for each $\mathbf{u} \in \mathrm{D}(\mathbf{x})$, then $\mathbf{x} \in \rho$, where $\mathrm{D}(\mathbf{x}) = \{\mathbf{u} \mid \mathbf{u} \in M^{(n-m)}$ and $\mathbf{x} = \mathbf{uv}$ for some $\mathbf{v} \in M^{(m)}\}$.

Examples 1.1. a) The set $\Delta^n(M) = \{x^n \mid x \in M^n\}$ (thin diagonal of M^n) is an (n,m)-equivalence and is a subset of any (n,k)-equivalence for any n,m,k.

b) The set $\Delta_1^3(M) = \{xxy \mid xxy \in M^{(3)}\}$ (thick diagonal of M^3 is a (3,1)-equivalence but it is not a (3,2)-equivalence, and if Δ_1^3 is a subset of a (3,2)-equivalence ρ, then $\rho = M^{(3)}$.

c) The set $\Delta_1^4(M) = \{xxxy \mid xxxy \in M^{(4)}\}$ is a (4,t)-equivalence for t=1,2.

d) The set $\Delta_2^4(M) = \{xxyy \mid xxyy \in M^{(4)}\}$ is a (4,t)-equivalence for t=1,2,3.

e) Let E be the Euclidean plane. The relation Coll$\subseteq E^{(3)}$ defined by: (A,B,C)$\in$Coll if and only if the points A,B,C are collinear, i.e. A,B,C are points on a line, is a (3,1)-equivalence that is not a (3,2)-equivalence. Moreover, $\Delta_1^3(E) \subseteq$Coll.

f) Let P be the Euclidean space. The relation Comp$\subseteq E^4$ defined by: (A,B,C,D)$\in$Comp if and only if the points A,B,C,D are coplanar, i.e. A,B,C,D are points on a plane, is a (4,1)-equivalence that is not a (4,2)-equivalence. Moreover, $\Delta_1^4(P) \subseteq$Comp, and $\Delta_2^4(P) \subseteq$Comp.

With the above notions, a (2,1)-equivalence is the usual notion of an equivalence relation. Any (2+1)-ary equivalence relation ρ as in D.1.2. is a (3,1)-equivalence on M such that $\Delta_1^3(M) \subseteq \rho$. Any (3+1)-ary equivalence relation ρ as in D.1.2. is a (4,1)-equivalence on M such that for any $xxyz \in M^{(4)}$, $xxyz \in \rho$, which implies that $\Delta_1^4(M) \subseteq \rho$ and $\Delta_2^4(M) \subseteq \rho$.

For small numbers n and m, (Tnm) has the following form:

(T21) For any $x, y, b \in M$, if $xb, by \in \rho$ then $xy \in \rho$, or using the usual notation for binary relation, if $x\rho b$ and $b\rho y$ then $x\rho y$;

(T31) For any $x, y, z, b \in M$, if $xyb, xbz, xyb \in \rho$, then $xyz \in \rho$;

(T32) For any $x, y, z, a, b \in M$, if $xab, ayb, abz \in \rho$, then $xyz \in \rho$;

(T41) For any $x, y, z, u, a, \in M$, if $xyza, xyau, xazu, ayzu \in \rho$, then $xyzu \in \rho$;

(T42) For any $x, y, z, u, a, b \in M$, if $xyab, xazb, xabu, ayzb, aybu, abzu \in \rho$ then $xyzu \in \rho$; and

(T43) For any $x, y, z, u, a, b, c \in M$, if $xabc, aybc, abzc, abcu \in \rho$, then $xyzu \in \rho$.

2. (4,2)-EQUIVALENCES

A characterization of (n,1)-equivalences and (n,n–1)-equivalences is given in [3]. The question of describing the (n,k)-equivalences for 1 < k < n–1 is open. We give same characterizations of (4,2)-equvalences on a set M.

Proposition 2.1. A subset $\rho \subseteq M^{(4)}$ is a $(4,2)$-equivalence on M if and only if there is a map $f : M^{(3)} \to \mathbf{B}(M)$, where $\mathbf{B}(M)$ is the Bulean of M, such that for any $x, y, u, v, a, b \in M$:

i) $x \in f(xxx)$;

ii) $x \in f(yuv)$ if and only if $y \in f(xuv)$;

iii) If $x \in f(aby)$ and $u, v \in f(abx) \cap f(aby)$, then $u \in f(abv)$ implies $v \in f(xyu)$; and

iv) $xyuv \in \rho$ if and only if $v \in f(xyu)$.

Proof: (a) Let ρ be a $(4,2)$-equivalence on M. We define the required map by $f(xyz) = \{u \mid xyzu \in \rho\}$. The fact that $\rho \subseteq M^{(4)}$ implies that the map f is well defined. The condition i) follows from (R4), i.e. the fact that $xxxx \in \rho$. If $x \in f(yuv)$ then $yuvx = xuvy \in \rho$, and so, $y \in f(xuv)$, i.e. ii) is satisfied. Next, let $x \in f(aby)$, $u, v \in f(abx) \cap f(aby)$ and $u \in f(abv)$. This implies that $xyab, xuab, xvab, yuab, yvab, uvab \in \rho$, and so, by (T42) we obtain that $xyuv \in \rho$, i.e. $v \in f(xyu)$. Hence, the condition iii) is satisfied. The condition iv) follows from the definition of the map f.

(b) Conversely, let $\rho \subseteq M^{(4)}$ and let $f : M^{(3)} \to \mathbf{B}(M)$ satisfies the condition i) – iv). Conditions i) and iv) imply that ρ satisfies (R4), i.e. $xxxx \in \rho$ for each $x \in M$. Next, let $xyab, xuab, xvab, yuab, yvab, uvab \in \rho$. Conditions ii) and iv) imply that $x \in f(aby)$, $u, v \in f(abx) \cap f(aby)$ and $u \in f(abv)$. All this, together with the condition iii) implies that $v \in f(xyu)$, and so, $xyuv \in \rho$, by condition iv). Hence ρ satisfies (R42), and so it is a $(4,2)$- equivalence.

Proposition 2.2. Let $\rho \subseteq M^{(4)}$ be a $(4,2)$-equivalence on M and let $\rho \cap \Delta_1^4(M) = \Delta^4(M)$. Then, $\overline{\rho} = \rho \cup \Delta_1^4(M)$ is a $(4,2)$-equivalence on M if and only if for any $x, y, u, v \in M$ we have:

(1) If $xxyu, xxyv, xxuv \in \rho$ and $yuv \notin \Delta^3(M)$ then $xyuv \in \rho$; and
(2) If $xxyu, xxyv, xyuv \in \rho$ then $xxuv \in \rho$.

Proof: (a) Let $\rho \cap \Delta_1^4(M) = \Delta^4(M)$, $\overline{\rho} = \rho \cup \Delta_1^4(M)$ and let ρ and $\overline{\rho}$ be a (4,2)-equivalence on M. If $xxyu, xxyv, xxuv \in \rho$ and $yuv \notin \Delta^3(M)$, then $\Delta_1^4(M) \subseteq \overline{\rho}$ and $\rho \subseteq \overline{\rho}$, imply that $xxxy, xxxu, xxxv, xxyu, xxyv, xxuv \in \overline{\rho}$. Since $\overline{\rho}$ is a (4,2)-equivalence, it follows that $xyuv \in \overline{\rho}$, and $yuv \notin \Delta^3(M)$ implies that $xyuv \notin \Delta_1^4(M)$. Hence, $xyuv \in \rho$. This proves that the condition (1) is satisfied. Next, let $xxyu, xxyv, xyuv \in \rho$. This, together with $\Delta_1^4(M) \subseteq \overline{\rho}$ implies that $xyxx, xyxu, xyxv, xyxu, xyxv, xyuv \in \overline{\rho}$. Since $\overline{\rho}$ is a (4,2)-equivalence, it follows that $xxuv \in \overline{\rho}$. Since $xxyu, xxyv \in \rho$ and $\rho \cap \Delta_1^4(M) = \Delta^4(M)$, it follows that $x \neq u$ and $x \neq v$, and so $xxuv \notin \Delta_1^4(M)$. Hence, $xxuv \in \rho$. This proves that the condition (2) is satisfied.

(b) Conversely, let ρ be a (4,2)-equivalence on M, $\rho \cap \Delta_1^4(M) = \Delta^4(M)$, and let the conditions (1) and (2) be satisfied. To check that $\overline{\rho}$ is a (4,2)-equivalences on M it is enough to check that $\overline{\rho}$ satisfies the condition (T42). Let $abxy, abxu, abxv, abyu, abyv, abuv \in \overline{\rho}$. If all $abxy, abxu, abxv, abyu, abyv$ and $abuv$, are in ρ, then, since ρ is a (4,2)-equivalence, $xyuv \in \rho$, and since $\rho \subseteq \overline{\rho}$, $xyuv \in \overline{\rho}$. Otherwise one of $abxy, abxu, abxv, abyu, abyv, abuv$ is not in ρ. Without loss of generality (w.l.o.g.) we may assume that $abxy \in \Delta_1^4(M)$, and again w.l.o.g. it is enough to check only the following two cases:

Case 1. $a = b = x \neq y$ and **Case 2.** $a = x = y \neq b$.

Case 1. In this case, $xxxy, xxxu, xxxv, xxyu, xxyv, xxuv \in \overline{\rho}$. If $x = u$ or $x = v$, then $xuyv = xxyv \in \overline{\rho}$ or $xvyu = xxyu \in \overline{\rho}$, i.e. $xyuv \in \overline{\rho}$. So, let $x \neq u$ and $x \neq v$. Then, since $x \neq y$, it follows that $xxyu, xxyv, xxuv \notin \Delta_1^4(M)$. This implies that $xxyu, xxyv, xxuv \in \rho$. If $yuv \notin \Delta^3(M)$, then the condition (1) implies that $xyuv \in \rho$. Otherwise, $y = u = v$, and $xyuv = xyyy \in \Delta_1^4(M) \subseteq \overline{\rho}$. All this shows that in this case, $xyuv \in \overline{\rho}$.

Case 2. In this case, $xxxb, xbxu, xbxv, xbuv \in \overline{\rho}$. If $xbxu, xbxv, xbuv \in \rho$, then the condition (2) implies that $xxuv \in \rho$, and so, $xyuv = xxuv \in \rho \subseteq \overline{\rho}$. Otherwise, one of $xbxu, xbxv, xbuv$ is not in ρ. If $xbxu \notin \rho$, since $x \neq b$ it follows that $x = u$, and then $xyuv = xxxv \in \Delta_1^4(M) \subseteq \overline{\rho}$, Similarly, if $x = v$, then $xyuv = xxxu \in \Delta_1^4(M) \subseteq \overline{\rho}$. If $x \neq u$ and $x \neq v$, then, since $x \neq b$, it follows that $xbuv \notin \rho$ and $b = u = v$. This implies that $xyuv = xxbv \in \overline{\rho}$. All this shows that in this case or $xvyu = xxyu \in \overline{\rho}$, i.e. $xyuv \in \overline{\rho}$. So, let $x \neq u$ and $x \neq v$. Then, since $x \neq y$, it follows that $xxyu, xxyv, xxuv \notin \Delta_1^4(M)$. This implies that $xxyu, xxyv, xxuv \in \rho$. If $yuv \notin \Delta^3(M)$, then the condition (1) implies that $xyuv \in \rho$. Otherwise, $y = u = v$, and $xyuv = xyyy \in \Delta_1^4(M) \subseteq \overline{\rho}$. All this shows that in this case, $xyuv \in \overline{\rho}$.

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On $(3,2,\rho)$-K-metrizable spaces

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Abstract: *Let M be a $(3,2,\rho)$-K-metrizable topological space. We show that: M is normal; if M is compact or has the Bolzano-Weierstrass property, then it has a Lebesgue number; and if M is* T_1 *space, then it is paracompact.*
Keywords: $(3,2,\rho)$*-metric, metric space, topology*

1. INTRODUCTION

The geometric properties, their axiomatic classification and the generalization of metric spaces have been considered in a lot of papers such as: K. Menger ([13]), V. Nemytzki, P.S. Aleksandrov ([15], [1]), Z. Mamuzic ([12]), S. Gäler ([10]), A.V. Arhangelskii, M. Choban, S. Nedev ([2], [4], [16]), B.C. Dhage, Z. Mustafa, B. Sims ([5], [14]), and many others. The notion of (n,m,ρ)-metric is introduced in [6]. Connections between some of the topologies induced by a $(3,1,\rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric are given in [7]. For a given $(3,j,\rho)$-metric d on a set M, $j\in\{1,2\}$, seven topologies $\tau(G,d),\tau(H,d),\tau(D,d),\tau(N,d)$, $\tau(W,d)$ and $\tau(K,d)$ on M, induced by d, are defined in [8], and several properties of these topologies are shown.

In this paper we are concerned with $(3,2,\rho)$-K-metrizable topological spaces (M,τ), i.e., with spaces (M,τ), with $\tau=\tau(K,d)$ for a $(3,2,\rho)$-metrics d on M. We define a Lebesgue number for a covering of $(3,2,\rho)$-K-metrizable topological spaces and prove the following results: a) Any $(3,2,\rho)$-K-metrizable topological space is normal; b) Any open covering of a $(3,2,\rho)$-K-metrizable topological space that is compact or has the Bolzano-Weierstrass property, has a Lebesgue number; and c) Any $(3,2,\rho)$-K-metrizable topological space that is T_1 is paracompact.

2. TOPOLOGIES INDUCED BY $(3,2,\rho)$ -METRIC

Let $M \neq \emptyset$. We denote by $M^{(3)}$ the symmetric third power of M, i.e., $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$(x,y,z)\alpha(u,v,w) \Leftrightarrow (u,v,w)$ is a permutation of (x,y,z).

Let $d : M^{(3)} \to [0,+\infty)$. We state two conditions for such a map:

(M0) $d(x,x,x) = 0$, for any $x \in M$;

(M1) $d(x,y,z) \leq d(x,a,b) + d(a,y,b) + d(a,b,z)$, for any $x,y,z,a,b \in M$.

For a map d as above let $\rho = \{(x,y,z) \mid (x,y,z) \in M^{(3)}, d(x,y,z) = 0\}$. The set ρ is a (3,2)-equivalence on M defined as in [6], and discussed in [7]. The set $\Delta = \{(x,x,x) \mid x \in M\}$ is a $(3,2)$ -equivalence on M.

Definition 2.1. If $d : M^{(3)} \to [0,+\infty)$ and ρ are as above and d satisfies (M0) and (M1) we say that d is a $(3,2,\rho)$**-metric** on M. If d is a $(3,2,\Delta)$ -metric on M, we say that d is $(3,2)$ **-metric** on M.

Example 2.1. Let $M \neq \emptyset$. The map $d : M^{(3)} \to [0,+\infty)$ defined by:

$d(x,y,z) \leq \begin{cases} 0, (x,y,z) \in \Delta \\ 1, (x,y,z) \notin \Delta \end{cases}$ is a $(3,2)$ -metric on M (**discrete** 3 **-metric**).

It is easy to check that if d is a $(3,2,\rho)$-metric on M, then for any $x,y,z \in M$, $d(x,x,y) \leq 2d(x,y,y)$ and $d(x,x,y) \leq 2d(x,z,z) + d(y,z,z)$.

Let d be a $(3,2,\rho)$-metric on M, $x \in M$ and $\varepsilon > 0$. We define, as in [8], a "little" ε-ball with centre at x and radius ε by $L(x,\varepsilon) = \{y \mid y \in M, d(x,y,y) < \varepsilon\}$.

Next, we define two topologies on M induced by d as follows:

1) $\tau(K,d)$ -the topology on M generated by all the ε-balls $L(x,\varepsilon)$;

2) $\tau(S,d)$ -the topology defined by:

$$U \in \tau(S,d) \Leftrightarrow (\forall x \in U)(\exists \varepsilon > 0)\ L(x,\varepsilon) \subseteq U.$$

Proposition 2.1. The set $\{L(x,\varepsilon) \mid x \in M, \varepsilon > 0\}$ is a base for $\tau(K,d)$ and $\tau(K,d) = \tau(S,d)$.

Proof. It is enough to show that $L(x,\varepsilon) \in \tau(S,d)$ for any $x \in M, \varepsilon > 0$. Let $y \in L(x,\varepsilon)$ and let $4\delta = \varepsilon - d(x,y,y)$. Then, for each $z \in L(y,\delta)$, $d(x,z,z) \leq d(x,y,y) + 2d(z,y,y) \leq d(x,y,y) + 4d(y,z,z) < d(x,y,y) + 4\delta$ and $d(x,y,y) < \varepsilon - 4\delta$ implies that $d(x,z,z) < \varepsilon$,i.e., $z \in L(y,\delta) \subseteq L(x,\varepsilon)$. All this shows that $L(x,\varepsilon) \in \tau(S,d)$.

Definition 2.2. We say that the topological space (M,τ) is $(3,2,\rho)$**-K-metrizable** if there is $(3,2,\rho)$-metric d such that $\tau=\tau(K,d)$.

3. SOME PROPERTIES OF $(3,2,\rho)$-K-METRIZABLE SPACES

Let (M,τ) be a $(3,2,\rho)$-K-metrizable topological space.

Proposition 3.1. For any $F\subseteq M$,

$$x\in\overline{F}\Leftrightarrow d(x,x,F)=\inf\{d(x,x,y)\mid y\in F\}=0.$$

Proof. Let $x\in\overline{F}$ and $d(x,x,F)=\varepsilon>0$. If there is a $z\in L(x,\varepsilon/2)\cap F$, then $d(x,x,z)\geq\varepsilon$ and $d(x,z,z)<\varepsilon/2$ imply $\varepsilon\leq d(x,x,z)\leq 2d(x,z,z)<\varepsilon$. Hence, $L(x,\varepsilon/2)\cap F=\emptyset$, but this contradicts the assumption that $x\in\overline{F}$.

Conversely, let $d(x,x,F)=0$. Then $(\forall\varepsilon>0)(\exists y\in F)$, $d(x,x,y)<\varepsilon/2$. This implies that $d(x,y,y)\leq 2d(x,x,y)<\varepsilon$. Therefore, $y\in L(x,\varepsilon)\cap F$. Since $L(x,\varepsilon)\cap F\neq\emptyset$ for any $\varepsilon>0$ it follows that $x\in\overline{F}$.

Definition 3.1. For any subset A of M and $\varepsilon>0$, we say that the set $L(A,\varepsilon)=\cup_{a\in A}L(a,\varepsilon)$ is the ε**-neighbourhood** of A. ($L(A,\varepsilon)$ is an open set as union of open sets.)

Proposition 3.2. For $A\subseteq M,\varepsilon>0$, $L(A,\varepsilon)=\{x\in M\mid d(x,x,A)<\varepsilon\}$.

Proof. Let $z\in L(A,\varepsilon)$ and let $z\in L(a,\varepsilon)$ for $a\in A$. Then, $d(z,z,a)<\varepsilon$ implies that $d(z,z,A)\leq d(z,z,a)<\varepsilon$, and so $z\in\{x\in M\mid d(x,x,A)<\varepsilon\}$.

Let $z\in\{x\in M\mid d(x,x,A)<\varepsilon\}$. Then $d(z,z,A)<\varepsilon$. This implies that there is an $a\in A$ such that $d(z,z,a)<\varepsilon$, i.e., $z\in L(a,\varepsilon)\subseteq L(A,\varepsilon)$.

Proposition 3.3. Let $F\subseteq M$ be a closed set. Then F is a G_δ – set.

Proof. Let $F\subseteq M$ be a closed set. For each $n\in\mathbf{N}$, $L(F,1/n)$ is open and $F\subseteq L(F,1/n)$. This implies that $F\subseteq\cap_{n=1}^{\infty}L(F,1/n)$. Next, we will show that $\cap_{n=1}^{\infty}L(F,1/n)\subseteq F$. Let $x\in L(F,1/n)$ for each $n\in\mathbf{N}$. For $n\in\mathbf{N}$, let $y_n\in F$ be such that $x\in L(y_n,1/n)$. Then $d(x,x,F)\leq d(x,x,y_n)<1/n$, for each $n\in\mathbf{N}$. This implies that $d(x,x,F)=0$. **Proposition 3.1.** implies that $x\in\overline{F}=F$ (F is closed). Hence, $\cap_{n=1}^{\infty}L(F,1/n)\subseteq F$.

Collorary 3.1. Let $U\subseteq M$ be an open set. Then U is an F_σ - set.

Proposition 3.4. For any $A\subseteq M$, the function $f_A:M\to[0,+\infty)$ defined by $f_A(x)=d(x,x,A)$ is continuous.

Proof. Let $x,y\in M$ and $a\in A$.Then:

$d(x,x,a) \le 2d(x,y,y) + d(a,y,y) \Rightarrow d(x,x,A) - d(y,y,A) \le 2d(x,y,y)$;
and
$d(y,y,a) \le 2d(y,x,x) + d(a,x,x) \Rightarrow d(x,x,A) - d(y,y,A) \ge -2d(x,x,y)$.

All this implies that:
$-4d(x,y,y) \le -2d(y,x,x) \le d(x,x,A) - d(y,y,A) \le 2d(x,y,y) \le 4d(x,y,y)$
, i.e., $|d(x,x,A) - d(y,y,A)| \le 4d(x,y,y)$.

Let $\varepsilon > 0$. If we choose $\delta = \varepsilon/4$, then $|f_A(x) - f_A(y)| < \varepsilon$ for every $y \in L(x,\delta)$. Hence, f_A is continuous function in $x \in M$.

Collorary 3.2. The space (M,τ) is normal.

Proof. Let A and B be two disjoint closed subsets of (M,τ). We define a function $f : M \to \mathbf{R}$ by $f(x) = f_A(x) - f_B(x)$. Since $f_A(x)$ and $f_B(x)$ are continuous, the function f is continuous. For $x \in B = \overline{B}$, $d(x,x,B) = 0$, and since $x \notin A = \overline{A}$, $d(x,x,A) > 0$, and so $f(x) > 0$. Hence, $B \subseteq U = f^{-1}(0,+\infty)$. Similarly, $A \subseteq V = f^{-1}(-\infty,0)$. The sets U and V are open and $U \cap V = \emptyset$.

Collorary 3.3. If the space (M,τ) is T_1, then it is T_4.

Deffinition 3.2. We say that a nonempty subset A of M is bounded if there exists an $r > 0$ such that $d(x,y,z) < r$ for every $x,y,z \in A$.

If A is bounded, the number $\sup\{d(x,y,z) \mid x,y,z \in A\}$ is called the **diameter** of A, and we write $\operatorname{diam} A = \sup\{d(x,y,z) \mid x,y,z \in A\}$.

If A is not bounded, we write $\operatorname{diam} A = \infty$.

Definition 3.3. Let $\boldsymbol{\omega}$ be an open covering of (M,τ). We say that a number $\delta > 0$ is a **Lebesgue number** for $\boldsymbol{\omega}$ if for each $B \subseteq M$ such that $\operatorname{diam} B < \delta$, there exists an element $A \in \boldsymbol{\omega}$ such that $B \subseteq A$.

Proposition 3.5. If (M,τ) is compact, then any open covering $\boldsymbol{\omega}$ of M has a Lebesgue number.

Proof. Let $\boldsymbol{\omega}$ be an open covering of M. If M is an element of $\boldsymbol{\omega}$, then each $\delta > 0$ is a Lebesgue number for $\boldsymbol{\omega}$. So, assume that M is not an element of $\boldsymbol{\omega}$. There exists a finite subcollection $\{A_1, A_2, ..., A_n\}$ of $\boldsymbol{\omega}$ that covers M. We define $f : M \to \mathbf{R}$ by $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x,x,M \setminus A_i)$.

For each $x \in M$, there is an $i \in \{1,2,...,n\}$ such that $x \in A_i$. Then the set $M \setminus A_i$ is closed, and from **Proposition 3.1.** it follows that $d(x,x,M \setminus A_i) > 0$. So, we have shown that $f(x) > 0$ for all $x \in M$. Since M is compact and f is continuous, f has a minimum value. Let $2\delta = \min\{f(x) \mid x \in M\} > 0$ and $B \subseteq M$ such that $\operatorname{diam} B < \delta$. For $y \in B$, $B \subseteq L(y,\delta)$ and $2\delta \le f(y) \le d(y,y,M \setminus A_m)$, where $d(y,y,M \setminus A_m)$ is the largest of the numbers $d(y,y,M \setminus A_i)$, $i \in \{1,2,...,n\}$. This implies that $B \subseteq L(y,\delta) \subseteq A_m$. The required Lebesgue number is δ.

Proposition 3.6. If (M,τ) has the Bolzano-Weierstrass property (each infinite subset of M has at least one point of accumulation), then each open covering of M has a Lebesgue number.

Proof. Let $\boldsymbol{\omega} = \{O_\alpha \mid \alpha \in \mathbf{I}\}$ be an open covering of the space (M,τ) with no Lebesgue number. Then for each $n \in \mathbf{N}$ there is a set B_n such that $\operatorname{diam} B_n < 1/n$ and $B_n \not\subset O_\alpha$ for each $\alpha \in \mathbf{I}$. Let $x_n \in B_n$ and $A = \{x_1, x_2, ..., x_n, ...\}$. Then $B_n \subseteq L(x_n, 1/n)$. If A is finite, then some point $x \in M$ occurs infinetly often in the sequence $x_1, x_2, ...$. Then there is a $\beta \in \mathbf{I}$ such that $x \in O_\beta$. Since O_β is open there is a $\delta > 0$ such that $L(x,\delta) \subseteq O_\beta$. We may choose an $n \in \mathbf{N}$ such that $1/n < \delta$ and $x_n = x$. So, $B_n \subseteq L(x_n, 1/n) = L(x, 1/n) \subseteq L(x,\delta) \subseteq O_\beta$, which is a contradiction.

If A is infinite, then A has at least one point of accumulation x. Then $x \in O_\beta$ for some $\beta \in \mathbf{I}$, and there is a $\delta > 0$ such that $L(x,\delta) \subseteq O_\beta$. There are infinitely many points of A in the neighbourhood $L(x,\delta/8)$ of x. We choose an $n \in \mathbf{N}$ such that $1/n < \delta/8$ and $x_n \in L(x,\delta/8)$. If $z \in L(x_n, 1/n)$, then $d(x,z,z) \le d(x,x_n,x_n) + 2d(z,x_n,x_n) \le d(x,x_n,x_n) + 4d(z,z,x_n) < \delta$, i.e., $z \in L(x,\delta)$. So, $B_n \subseteq L(x_n, 1/n) \subseteq L(x,\delta) \subseteq O_\beta$, which is again a contradiction.

Proposition 3.7. If (M,τ) is T_1 space, then it is paracompact.

Proof. We will prove that (M,τ) is paracompact by proving that there exists a sequence $\{\boldsymbol{\omega}_n \mid n \in \mathbf{N}\}$ of open coverings that is locally starring for every open covering of M. (Arhangel'skii theorem, see e.g. [9].)

Let $\boldsymbol{\omega}_n = \{L(x,1/n) \mid x \in M\}$ and let $\{U_\alpha \mid \alpha \in \mathbf{I}\}$ be an open covering of M. For any $x \in M$ there is an $\alpha \in \mathbf{I}$ such that $x \in U_\alpha$. Then there is an $n \in \mathbf{N}$ such that $L(x,1/n) \subseteq U_\alpha$. Let $U = L(x,1/3n)$.

For $z \in St(U, \boldsymbol{\omega}_{12n}) = \bigcup\{L(y,1/12n) \mid L(y,1/12n) \cap U \neq \emptyset\}$, there are $y, a \in M$ such that $z \in L(y,1/12n)$ and $a \in L(y,1/12n) \cap U$. Then

$$\begin{aligned} d(x,z,z) &\le d(x,y,y) + 2d(z,y,y) \le d(x,a,a) + 2d(y,a,a) + 4d(y,z,z) \\ &< 1/3n + 2\cdot 1/12n + 4\cdot 1/12n < 1/3n + 1/3n + 1/3n = 1/n. \end{aligned}$$

So, $z \in L(x,1/n)$, i.e., $St(U, \boldsymbol{\omega}_{12n}) \subseteq L(x,1/n) \subseteq U_\alpha$.

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Proximate Fundamental Group

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***Abstract**: The intrinsic definition of shape based on proximate sequences (f_n) for compact metric spaces and for paracompact spaces based on proximate nets ($f_{\mathcal{V}}$)* indexed by open coverings are presented in [3] and [4]. *Using this approach we define proximate fundamental group, an invariant of shape of a space.*

***Keywords:** $\mathcal{U}$ - path, homotopy up to a covering, $\mathcal{U}$ - continuous loop, proximate loop, proximate fundamental group.*

1. INTRODUCTION

In this paper we will use the techniques for giving the intrinsic definition of shape based on proximate sequences for compact metric spaces introduced in [3] and [4] to define proximate fundamental group, an invariant of shape of a space. The results are more generalized and applicable to all topologic spaces since we will use proximate nets indexed by open coverings. After defining proximate fundamental group at the end of this paper our new challenge will be to define a proximate functor as new approach for determining the shape of the paracompact topologic space.

2. HOMOTOPY OF $\mathcal{U}$ - PATHS

Let X is topological space and $I=[0,1]$.

Definition 2.1: Let $\mathcal{U}$ is an open covering of the space X and $x_0, x_1 \in X$ are fixed points. The $st(\mathcal{U})$ - continuous function $k_{\mathcal{U}} : I \to X$ such that is $\mathcal{U}$ - continuous on $\partial I = \{0,1\}$ and $k_{\mathcal{U}}(0) = x_0, k_{\mathcal{U}}(1) = x_1$ is called $\mathcal{U}$ - path with endpoints x_0 and x_1.

Definition 2.2: Let $\mathcal{U}$ is an open covering of the space X and $k_{\mathcal{U}}, l_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - paths with endpoints x_0 and x_1. We say that the $\mathcal{U}$

- paths k_{U} and l_{U} are U - homotopic paths relatively endpoints, if there exists a function $F: I\times I \to X$ such that:

(I) F is $st^{2}(U)$ - continuous;

(II) F is $st(U)$ - continuous on $\partial I^{2} = \partial(I\times I)$;

(III) F is U - continuous on $\partial I\times\partial I = \{(0,0),(0,1),(1,0),(1,1)\}$;

and satisfies the usual conditions for homotopy of paths relatively endpoints

(IV) $F(t,0) = k_{U}(t)$ and $F(t,1) = l_{U}(t)$ for all points $t\in I$;

(V) $F(0,s) = k_{U}(0) = l_{U}(0) = x_{0}$ and $F(1,s) = k_{U}(1) = l_{U}(1) = x_{1}$ for all elements $s\in I$.

When two U - paths k_{U} and l_{U} with same endpoints are U - homotopic relatively endpoints we denote by $k_{U} \underset{U}{\sim} l_{U}\left(rel\{0,1\}\right)$, i.e $k_{U} \underset{U}{\sim} l_{U}$.

Proposition 2.1: The relation of U - homotopy $k_{U} \underset{U}{\sim} l_{U}\left(rel\{0,1\}\right)$ of U - paths is an equivalence relation.

The homotopy class of U - paths, $k_{U}: I \to X$ we will denote by $[k_{U}]$.

Let consider an open covering U of the space X, and two U - paths $k_{U}, l_{U}: I \to X$ such that $k_{U}(1) = l_{U}(0)$. We define a composition by:

$$(k_{U} * l_{U})(t) = \begin{cases} k_{U}(2t), & 0 \le t \le \dfrac{1}{2} \\ l_{U}(2t-1), & \dfrac{1}{2} \le t \le 1 \end{cases}$$

By Theorem 2.2 [3] this composition is well defined and $st(U)$ - continuous function. Also by the definition of U - paths $k_{\mathcal{U}}, l_{\mathcal{U}}: I \to X$ the juxtaposition $k_{U} * l_{U}$ is U - continuous on $\partial I = \{0,1\}$. So $k_{U} * l_{U}$ is U - path.

Theorem 2.1: Let $k_{U}^{0}, k_{U}^{1}: I \to X$, $l_{U}^{0}, l_{U}^{1}: I \to X$ are U - paths such that $k_{U}^{0} \underset{U}{\sim} k_{U}^{1}\left(rel\{0,1\}\right)$, $l_{U}^{0} \underset{U}{\sim} l_{U}^{1}\left(rel\{0,1\}\right)$ and the compositions $k_{U}^{0} * l_{U}^{0}$ and $k_{U}^{1} * l_{U}^{1}$ are defined. Then $k_{U}^{0} * l_{U}^{0} \underset{U}{\sim} k_{U}^{1} * l_{U}^{1}\left(rel\{0,1\}\right)$.

Proof: Because $k^0_{\mathrm{U}} \underset{\mathrm{U}}{\sim} k^1_{\mathrm{U}} \left(rel\{0,1\}\right)$ and $l^0_{\mathrm{U}} \underset{\mathrm{U}}{\sim} l^1_{\mathrm{U}} \left(rel\{0,1\}\right)$ there exist U - homotopies $K: I\times I \to X$ and $L: I\times I \to X$ between the U - paths k^0_{U}, k^1_{U} and l^0_{U}, l^1_{U}, respectively.

Let define a function $H: I\times I \to X$ by:

$$H(t,s) = \begin{cases} K(2t,s), & 0 \le t \le \dfrac{1}{2} \\ L(2t-1,s), & \dfrac{1}{2} \le t \le 1 \end{cases}$$

By Theorem 2.2 [3] is $st^2(\mathrm{U})$ - continuous since K and L are $st^2(\mathrm{U})$ - continuous, K and L are $st(\mathrm{U})$ - continuous on ∂I^2.

By the definition of the function H and the facts that K and L are $st(\mathrm{U})$ - continuous on ∂I^2 follows that the function H is $st(\mathrm{U})$ - continuous on ∂I^2. Also, considering the definition of the function H since K and L are U - continuous in the points $(0,0),(0,1),(1,0),(1,1)$ then the function H is U - continuous in the points. The usual conditions for homotopy H of paths relatively endpoints (IV) and (V) can be checked in the usual way (for example as in [5]). In this way, we showed that $k^0_{\mathrm{U}} * l^0_{\mathrm{U}} \underset{\mathrm{U}}{\sim} k^1_{\mathrm{U}} * l^1_{\mathrm{U}} \left(rel\{0,1\}\right)$.

Theorem 2.2: Let $k_{\mathrm{U}}, l_{\mathrm{U}}, p_{\mathrm{U}} : I \to X$ are U - paths in X and compositions $k_{\mathrm{U}} * l_{\mathrm{U}}$ and $l_{\mathrm{U}} * p_{\mathrm{U}}$ are defined, $k_{\mathrm{U}}(1) = l_{\mathrm{U}}(0)$ and $l_{\mathrm{U}}(1) = p_{\mathrm{U}}(0)$. Then $(k_{\mathrm{U}} * l_{\mathrm{U}}) * p_{\mathrm{U}} \underset{\mathrm{U}}{\sim} k_{\mathrm{U}} * (l_{\mathrm{U}} * p_{\mathrm{U}}) \left(rel\{0,1\}\right)$.

Proof: First let represent the square $I\times I$ as union of three closed sets A, B and C, i.e $I\times I = A\cup B\cup C$, where $A = \left\{ (t,s) \mid s\in I, 0 \le t \le \dfrac{s+1}{4} \right\}$, $B = \left\{ (t,s) \mid s\in I, \dfrac{s+1}{4} \le t \le \dfrac{s+2}{4} \right\}$, $C = \left\{ (t,s) \mid s\in I, \dfrac{s+2}{4} \le t \le 1 \right\}$. We consider the following functions $a: A \to X$, $b: B \to X$ and $c: C \to X$:

$a(t,s) = k_{\mathrm{U}} \circ f(t,s)$ where $f(t,s) = \dfrac{4t}{s+1}$; $a(t,s) = k_{\mathcal{U}}\left(\dfrac{4t}{s+1} \right)$,

$b(t,s) = l_{\mathrm{U}} \circ g(t,s)$, where $g(t,s) = 4t-1-s$; $b(t,s) = l_{\mathcal{U}}(4t-1-s)$,

$c(t,s) = p_{\mathrm{U}} \circ h(t,s)$; where $h(t,s) = \dfrac{4t-2-s}{2-s}$; $c(t,s) = p_{\mathcal{U}}\left(\dfrac{4t-2-s}{2-s}\right)$.

Now define a function $H : I \times I \to X$ by:

$$H(t,s) = \begin{cases} a(t,s), & (t,s) \in A \\ b(t,s), & (t,s) \in B \\ c(t,s), & (t,s) \in C \end{cases}$$

The functions f, g and h defined on A, B and C respectively, are continuous. The U - paths k_{U}, l_{U} and p_{U} are $st(\mathrm{U})$ - continuous. Therefore, $a = k_{\mathrm{U}} \circ f$ is st(U) – continuous. Similarly, b and c are $st(\mathrm{U})$ - continuous. Since a, b and c are $st^2(\mathrm{U})$ - continuous, a and b are $st(\mathrm{U})$ - continuous on $A \cap B$ and b and c are $st(\mathrm{U})$ - continuous on $B \cap C$, then by Theorem 2.2 [3], it follows that H is $st^2(\mathrm{U})$ - continuous. By the definition of the function H and the fact that a, b and c are $st(\mathrm{U})$ - continuous on ∂A, ∂B and ∂C, and U - continuous on the vertices of A, B and C, it follows that the function H is $st(\mathrm{U})$ - continuous on ∂I^2.) The continuous functions f, g and h defined on A, B and C respectively, map vertices of A, B and C to $\partial I = \{0,1\}$. The U - paths k_{U}, l_{U} and p_{U} are U - continuous on $\partial I = \{0,1\}$. Therefore, $a = k_{\mathrm{U}} \circ f$ is U - continuous function on the vertices of A, b is U - continuous function on the vertices of B, and c is U - continuous functions on the vertices of C. Considering the definition of the function H since a is U - continuous on $(0,0),(0,1)$, and c is U - continuous on $(1,0),(1,1)$, we conclude that the function H is U - continuous on $(0,0),(0,1),(1,0),(1,1)$.

The usual conditions for homotopy H of paths relatively endpoints (IV) and (V) can be checked in the usual way (for example as in [5]).

Therefore we showed that $(k_{\mathrm{U}} * l_{\mathrm{U}}) * p_{\mathrm{U}} \underset{\mathrm{U}}{\sim} k_{\mathrm{U}} * (l_{\mathrm{U}} * p_{\mathrm{U}})(rel\{0,1\})$.

Let X is a topological space and $x_0 \in X$. The function $c_{x_0} : I \to X$ defined by $c_{x_0}(t) = x_0$, for all $t \in I$, is constant path and by prop 1.3 (i) [1] is $\mathcal{U}$ - continuous on the set I. Therefore, the function $c_{x_0} : I \to X$ is $\mathcal{U}$ - path, and we wil call it **constant $\mathcal{U}$ - path**.

By similar reasoning from Theorem 2.2 and techniques used in [5] from the topic Fundamental group it is easy to show the following theorems:

Theorem 2.3: Let $k_{\mathcal{U}} : I \to X$ is $\mathcal{U}$ - path with endpoints x_0 and x_1. Then $k_{\mathcal{U}} * c_{x_1} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} \left(rel\{1,0\} \right)$ and $c_{x_0} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} \left(rel\{1,0\} \right)$.

Definition 2.3: Let X is topologic space and $k_{\mathcal{U}} : I \to X$ is $\mathcal{U}$ - path in X. The $\mathcal{U}$ - path in X $k_{\mathcal{U}}^{-1} : I \to X$ defined by $k_{\mathcal{U}}^{-1}(t) = k_{\mathcal{U}}(1-t)$ is called inverse $\mathcal{U}$ - path of the $\mathcal{U}$ - path $k_{\mathcal{U}}$. Notice that $(k_{\mathcal{U}}^{-1})_{\mathcal{U}}^{-1} = k_{\mathcal{U}}$.

Theorem 2.4: Let $k_{\mathcal{U}}, l_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - paths in X such that $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \left(rel\{0,1\} \right)$ then $k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}^{-1} \left(rel\{0,1\} \right)$.

Theorem 2.5: Let $k_{\mathcal{U}} : I \to X$ is $\mathcal{U}$ - path in X such that $k_{\mathcal{U}}(0) = x_0$ and $k_{\mathcal{U}}(1) = x_1$. Then is true that $k_{\mathcal{U}} * k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} c_{x_o} \left(rel\{0,1\} \right)$

3. PROXIMATE FUNDAMENTAL GROUP

We will consider a directed set $Cov(X)$, which consists of coverings of a space X. Usually, if X is compact metric space, $Cov(X)$ consists of all finite coverings. If X is paracompact space we can take $Cov(X)$ to consist of all locally finite coverings.

Definition 3.1: Let $\mathcal{U}$ is an open covering of the space X and $x_0 \in X$ is a fixed point. The $\mathcal{U}$ - path $k_{\mathcal{U}} : I \to X$ such that $k_{\mathcal{U}}(0) = k_{\mathcal{U}}(1) = x_0$ is called $\mathcal{U}$ - loop in x_0. The homotopy class of $\mathcal{U}$ - loops in x_0, $k_{\mathcal{U}} : I \to X$ we will denoted by $[k_{\mathcal{U}}]_{x_0}$.

Definition 3.2: A proximate loop in x_0 (over $Cov(X)$) is an indexed family $\underline{k} = \left(k_{\mathcal{U}} \middle| \mathcal{U} \in Cov(X) \right)$ such that $k_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} \left(rel\{0,1\} \right)$ for all $\mathcal{V} \prec \mathcal{U}$.

Definition 3.3: Two proximate loops $\underline{k}$ and $\underline{l}$ in x_0 are said to be homotopic up to coverings if $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \left(rel\{0,1\} \right)$ for all $\mathcal{U} \in Cov(X)$, we denote that by $\underline{k} \sim \underline{l} \left(rel\{0,1\} \right)$.

Proposition 3.1: The relation $\underline{k} \sim \underline{l} \left(rel\{0,1\} \right)$ is an equivalence relation. The homotopy class of proximate loop $\underline{k}$ in x_0 denoted by $[\underline{k}]_{x_0}$.

We consider the following set:

$prox\pi_1(X, x_0) = \left\{ [\underline{k}]_{x_0} \mid \underline{k} \text{ is proximate loop in } x_0 \right\}$.

In this set we define an operation $*$ by: $[\underline{k}]_{x_0} * [\underline{l}]_{x_0} = [\underline{k} * \underline{l}]_{x_0}$, where $\underline{k} * \underline{l}$ is defined as: $\underline{k} * \underline{l} = \left(k_{\mathcal{U}} * l_{\mathcal{U}} \mid \mathcal{U} \in Cov(X) \right)$.

We will show that this operation is well defined.

First we will find that $\underline{k} * \underline{l}$ is proximate loop in x_0. By the definition of the composition of two $\mathcal{U}$ - loops for all $\mathcal{U} \in Cov(X)$ the function $k_{\mathcal{U}} * l_{\mathcal{U}}$ is $\mathcal{U}$ - loop in x_0. Now, let consider any $\mathcal{V} \prec \mathcal{U}$. Since $\underline{k}$ and $\underline{l}$ are $\mathcal{U}$ - loops then $k_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} \left(rel\{0,1\} \right)$ and $l_{\mathcal{V}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \left(rel\{0,1\} \right)$, so by prop 1.3 (iii) [1] and Theorem 2.1 is true that $k_{\mathcal{V}} * l_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} * l_{\mathcal{U}} \left(rel\{0,1\} \right)$. Therefore, $\underline{k} * \underline{l}$ is proximate loop in x_0. Now, by Theorem 2.1 if $k_{\mathcal{U}}^0, k_{\mathcal{U}}^1 : I \to X$, $l_{\mathcal{U}}^0, l_{\mathcal{U}}^1 : I \to X$ are $\mathcal{U}$ - loops in x_0 such that $k_{\mathcal{U}}^0 \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}^1 \left(rel\{0,1\} \right)$, $l_{\mathcal{U}}^0 \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}^1 \left(rel\{0,1\} \right)$ then is true that $k_{\mathcal{U}}^0 * l_{\mathcal{U}}^0 \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}^1 * l_{\mathcal{U}}^1 \left(rel\{0,1\} \right)$.

Therefore, the operation $*$ in the set $prox\pi_1(X, x_0)$ is well defined.

By Theorems 2.2, 2.3, 2.4 and 2.5 we obtain the following theorem:

Theorem 3.1: The set $prox\pi_1(X, x_0)$ with the operation $*$ is a group. This group $prox\pi_1(X, x_0)$ **is called proximate fundamental group**.

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INTRINSIC SHAPE BASED ON ε-CONTINUITY AND ON CONTINUITY UP TO A COVERING ARE EQUIVALENT (II)

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Abstact: *In this paper it is shown how to a proximate sequence (f_n) from [2] to associate a proximate net from [3] . The function f_n from a proximate sequence (f_n)*, have to be $\mathcal{V}_n$ *- continuous*, for some sequence of coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$ cofinal in the all coverings, while *the function f_n from a proximate net (f_n)*, have to be ε_n *- continuous* some decreasing sequence (ε_n) tending to zero. *In [1], this result enables to prove an isomorphism between shape categories of Shekutkovski [2] and of Sanjurjo [3], both defined in intrinsic way.*

Keywords: *intrinsic definitions of shape, proximate sequence, proximate nets, isomorphism*

1. INTRODUCTION

Let X and Y be compact metric spaces. We repeat the intrinsic approaches to shape from [2] and [3]. According to [2] we have:

Definition 1. A function $f : X \to Y$ is $\mathcal{V}$-continuous, where $\mathcal{V}$ is finite covering of Y, if for every point $x \in X$ there exists a neighborhood U_x of x, and $V \in \mathcal{V}$, such that $f(U_x) \subseteq V$.

Definition 2. The functions $f, g : X \to Y$ are $\mathcal{V}$-homotopic, if there exists a continuous function $F : X \times I \to Y$ such that for every $x \in X$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$, F is $\mathrm{st}\mathcal{V}$-continuous and $F|_{N \times X}$ is $\mathcal{V}$– continuous for some neighbourhood $N = [0,\varepsilon) \cup (1-\varepsilon,1]$ of ∂I.

Definition 3. A cofinal sequence of finite covering $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$ is a sequence of finite covering of spaces, such that for any covering $\mathcal{V}$, there exists n, such that $\mathcal{V}_n \prec \mathcal{V}$.

Definition 4. The sequence (f_n) of functions $f_n : X \to Y$ is a proximate sequence from X to Y, if there exists a cofinal sequence of finite coverings of Y, $V_1 \succ V_2 \succ \ldots$, and for all indexes $m \ge n$, f_n and f_m are V_n-homotopic.

Definition 5. Two proximate sequences (f_n) and (f_n') are homotopic if there exists a cofinal sequence of finite coverings of Y, $V_1 \succ V_2 \succ \ldots$, such that (f_n) and (f_n') are V_n–homotopic for all integers n.

Let $(f_n): X \to Y$ be a proximate sequence over (V_n) and $(g_k): Y \to Z$ be a proximate sequence over (W_n). For a covering W_k of Z, there exists a covering V_{n_k} of Y such that $g(V_{n_k}) \prec W_{n_k}$. Then, the composition is the proximate sequence $(h_k) = (g_k f_{n_k}) : X \to Z$. In [2] is proven that compact metric spaces and homotopy classes of proximate sequences $[(f_n)]$ form the shape category InSh.

According to [3] we have:

Definition 6. A function $f : X \to Y$ is ε-continuous, if for every $x \in X$ there is a neighborhood of x whose image lies in the ε-neighborhood of the image of x.

Definition 7. The functions $f, g : X \to Y$ are ε-homotopic, if there exists a ε-continuous function $F : X \times I \to Y$ such that for every $x \in X$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

The relation of ε-homotopy is an equivalence relation on the set of ε-continuous functions.

Definition 8. A proximate net, from X to Y, is a sequence of (not necessarily continuous) functions $f_n : X \to Y$ such that for every $\varepsilon > 0$ there is an index n_0 such that f_n is ε-homotopic to f_{n+1} for every $n \ge n_0$. We denote proximate nets with $(f_n): X \to Y$, or just with (f_n).

Definition 9. Two proximate nets (f_n) and (f'_n) are homotopic if for every $\varepsilon > 0$, f_n is ε-homotopic to f'_n for almost every n.

We use symbol [ ] to denote homotopy classes.

Definition 10. A null sequence $\varepsilon_1 \ge \varepsilon_2 \ge \ldots \ge \varepsilon_n \ge \ldots$ of positive numbers is sequence of positive numbers such that $\varepsilon_n \to 0$ when $n \to \infty$.

Let $[(f_n)] : X \to Y$ and $[(g_n)] : Y \to Z$ be classes of proximate nets, and $(f_n): X \to Y$ and $(g_n): Y \to Z$ be their representatives. We choose a null sequence of positive numbers $\varepsilon_1 \ge \varepsilon_2 \ge \ldots \ge \varepsilon_n \ge \ldots$ such that g_n is

$\frac{\varepsilon_{n_0}}{2}$ homotopic to g_{n_0} for every $n \geq n_0$, null sequence $\delta_1 \geq \delta_2 \geq \ldots \delta_n \geq \ldots$ such that $d(g_n(y), g_n(y')) < \varepsilon_n$ for every δ_n-close points y и y' in Y, and sequence of indices $k_1 < k_2 < \ldots k_n < \ldots$ such that f_k is $\frac{\delta_n}{2}$-homotopic to f_{k_n} for every $k \geq k_n$. We define $[(g_n)][(f_n)] = [(g_n)(f_{k_n})]$. In [3] it is proven that in this way, we get a category denoted with NH , which is isomorphic to Borsuk shape category.

2. ε-CONTINUITY AND CONTINUITY UP TO A COVERING

In [1], it is shown that the category InSh and NH are isomorphic. We will complete the article [1], showing the unproved claims of that article.

Next three propositions give the connections between two concepts of ε- continuity.

Definition 11: ([5]) Function $f : X \to Y$ is ε- continuous in the sense of Klee, if there exists $\delta > 0$ such that for every $x, x' \in X$ for which $d(x, x') < \delta$, it follows $d(f(x), f(x')) < \varepsilon$.

Proposition 1. Let X and Y be compact metric spaces. If $f : X \to Y$ is ε-continuous function, then f is 2ε-continuous in the sense of Klee.

Proof. Let f is ε-continuous function, but is not 2ε- continuous in the sense of Klee. It follows $\forall \delta > 0$, $\exists x, x' \in X$ for which $d(x, x') < \delta$ and

$$d(f(x), f(x')) \geq 2\varepsilon. \qquad (1)$$

Specially for $\delta = \frac{1}{n}$ there exists $x_n, x_n' \in X$ for which $d(x_n, x'_n) < \frac{1}{n}$ and $d(f(x_n), f(x'_n)) \geq 2\varepsilon$.

Because X is compact metric space, there exists subsequences (x_{n_k}) and (x'_{n_k}) such that $x_{n_k} \to x_0$ and $x'_{n_k} \to x'_0$ for some $x_0, x'_0 \in X$. But, than $d(x_{n_k}, x'_{n_k}) \to d(x_0, x'_0)$. Because $d(x_{n_k}, x'_{n_k}) \to 0$ we have $d(x_0, x'_0) = 0$ or $x_0 = x'_0$.

Because f is ε-continuous, there exist neighborhood U of x_0, which image lies in the ε neighborhood of image of x_0. Because $x_{n_k}, x'_{n_k} \to x_0$

there exist k_0 such that $x_{n_k}, x'_{n_k} \in U$ for all $k \geq k_0$. Than $d(f(x_{n_k}), f(x_0)) < \varepsilon$ and $d(f(x'_{n_k}), f(x_0)) < \varepsilon$ for all $k \geq k_0$. Hence

$$d(f(x_{n_k}), f(x'_{n_k})) \leq d(f(x_{n_k}), f(x_0)) + d(f(x'_{n_k}), f(x_0)) < \varepsilon + \varepsilon = 2\varepsilon .$$

The last inequality contradicts inequality (1). ■

Proposition 2. Let X and Y be compact metric spaces. If $f : X \to Y$ is ε-continuous in the sense of Klee then f is ε-continuous function.

Proof. Let function $f : X \to Y$ be ε- continuous in the sense of Klee, and $x \in X$. It follows that there exists $\delta > 0$ such that for every $x' \in X$ for which $x' \in T_\delta(x)$, implies that $f(x') \in T_\varepsilon(f(x))$. This means that f is ε-continuous. ■

The next statement is given in [4], without a proof. It allows us to fix finite covering with good properties. In [4] is used notion ε-continuous function , but it means, ε-continuous function in the sense of Klee. We will prove it.

Proposition 3. Let X and Y be compact metric spaces. $f : X \to Y$ is ε-continuous function in the sense of Klee for some $\varepsilon > 0$, if and only if it is V-continuous for some finite cover V_ε consisting of open balls of radius ε.

Proof. Let f be ε-continuous function in the sense of Klee. Then there exists $\delta > 0$ such that for every $x, x' \in X$ for which $d(x, x') < \delta$, it follows $d(f(x), f(x')) < \varepsilon$. Let $T_\delta(x)$ denote open ball with center x, and radius ε. For fixed $x \in X$ and an arbitrary $x' \in T_\delta(x)$, it follows that $f(x') \in T_\varepsilon(f(x))$.

Since $\{T_\delta(x) \| x \in X\}$ is a covering for compact metric space X, there exists a finite subcovering $T_\delta(x_1), T_\delta(x_2), \ldots, T_\delta(x_n)$. Then $T_\varepsilon(f(x_1)), T_\varepsilon(f(x_2)), \ldots, T_\varepsilon(f(x_n))$ is a covering for $f(x)$. We are considering the family of open balls $\{T_\varepsilon(y) \| y \in Y\}$, which cover Y. There exists a finite subcovering which covers Y. We add open balls $T_\varepsilon(f(x_1)), T_\varepsilon(f(x_2)), \ldots, T_\varepsilon(f(x_n))$ and get finite covering V_ε of Y. Now if $x \in X$, it follows that there exists a natural number i∈{1,…,n}, such that $x \in T_\delta(x_i)$ and $f(T_\delta(x_i)) \subseteq T_\varepsilon(f(x_i)) \in \mathrm{V}_\varepsilon$.

Conversely, let f be $\mathcal{V}$-continuous for some finite cover $\mathcal{V}$ consisting of open balls of radius ε.

It follows that for every $x \in X$, there exists a neighborhood U_x on x, such that $f(U_x) \subseteq V \in \mathcal{V}$. There exists a $\delta > 0$ such that $T_\delta(x) \subseteq U_x$. $\{T_{\delta/2}(x) \mid x \in X\}$ is covering for compact metric space X, and it follows that there exists a finite subcovering $T_{\delta/2}(x_1), T_{\delta/2}(x_2), \ldots, T_{\delta/2}(x_n)$.

If $y, y' \in X$ and $d(y, y') < \dfrac{\delta}{2}$, then $y \in T_{\frac{\delta}{2}}(x_i)$ for some $i \in \{1, \ldots, n\}$. It follows that $d(y, x_i) < \dfrac{\delta}{2}$. Then $d(y', x_i) < d(y', y) + d(y, x_i) < \dfrac{\delta}{2} + \dfrac{\delta}{2} = \delta$.

We got that $y, y' \in T_\delta(x_i)$, from where we get that $f(x)$ and $f(x')$ belongs to some open ball with radius ε.■

By Lebesgue number lemma, if X is compact metric space, and $\mathcal{V}$ is covering of X, then there exists a number $\delta > 0$ such that, every subset of X having diameter less than δ, is contained in some member of the covering. The number δ is called a Lebesgue number of this covering.

From Lebesgue number lemma, it follows that, if f is $\mathcal{V}$-continuous function, than f is $\mathcal{V}_\varepsilon$-continuous function, for some finite cover $\mathcal{V}_\varepsilon$ consisting of open balls of radius ε.

If f is $st(\mathcal{V}_\varepsilon)$-continuous function, for some finite cover $(\mathcal{V}_\varepsilon)$ consisting of open balls of radius ε, than f is $\mathcal{V}_{3\varepsilon}$-continuous function, where $(\mathcal{V}_{3\varepsilon})$ is consisting of open balls with the same center as balls in $(\mathcal{V}_{3\varepsilon})$ and radius 3ε.

3. PROXIMATE SEQUENCES AND PROXIMATE NETS

Proposition 4. Let (f_n) is proximate sequence from X to Y over $(\mathcal{V}_n)$. Than tnere exist proximate subsequence (f_{n_k}) from X to Y over cofinal sequence of finite coverings $(\mathcal{V}_{\varepsilon_k})$, where coverings $\mathcal{V}_{\varepsilon_k}$ are consisting of ε_k balls and $\varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq \varepsilon_n \geq \ldots$ is null sequence from positive numbers.

Proof. Let (f_n) be proximate sequence from X to Y over $(\mathrm{V_n})$ and ε_1 is positive number. Let covering $(\mathrm{V}_{\varepsilon_1})$ of Y, consists of open balls with radius ε_1. Because Y is compact metric space, $(\mathrm{V}_{\varepsilon_1})$ can be chosen to be finite. Because $(\mathrm{V_n})$ is cofinal sequence, it follows there exist V_{n_1} such that $\mathrm{V}_{n_1} \prec \mathrm{V}_{\varepsilon_1}$. From Lebesgue number lemma there exist $\varepsilon_2 > 0$, $\varepsilon_2 \leq \frac{\varepsilon_1}{2}$ such that every open ball of radius ε_2, is contained in some set of covering $\mathrm{V}_{\varepsilon_1}$. Let $\mathrm{V}_{\varepsilon_2}$ be finite covering of Y which is consisting of ε_2 balls. Because $(\mathrm{V_n})$ is cofinal sequence, there exists a covering $(\mathrm{V}_{\mathrm{n}_2})$, $n_2 > n_1$ such that $\mathrm{V}_{n_2} \prec \mathrm{V}_{\varepsilon_2}$. Also $\mathrm{V}_{\varepsilon_2} \prec \mathrm{V}_{\varepsilon_1}$.

In k-th step from Lebesgue number lemma we get that there exist number $\varepsilon_k > 0$, $\varepsilon_k \leq \frac{\varepsilon_{k-1}}{2}$ such that every open ball of radius ε_k, is contained in some set of covering $\mathrm{V}_{\varepsilon_{k-1}}$. Let $\mathrm{V}_{\varepsilon_k}$ be finite covering of Y which is consisting of ε_k balls. Because $(\mathrm{V_n})$ is cofinal sequence, it follows that there exist covering $\mathrm{V}_{\mathrm{n_k}}$, $n_k > n_{k-1}$ such that $\mathrm{V}_{\mathrm{n_k}} \prec \mathrm{V}_{\mathrm{n_{k-1}}}$. Also $\mathrm{V}_{\varepsilon_k} \prec \ldots \prec \mathrm{V}_{\varepsilon_2} \prec \mathrm{V}_{\varepsilon_1}$.

In this way, we get a cofinal sequence of finite coverings $(\mathrm{V}_{\varepsilon_k})$ and a proximate sequence (f_{n_k}) of $\mathrm{V}_{\varepsilon_k}$ continuous functions $f_{n_k} : X \to Y$ over $(\mathrm{V}_{\varepsilon_k})$. Namely because for $n_k \leq n_l$, f_{n_k} and f_{n_l} are $\mathrm{V}_{\mathrm{n_k}}$ homotopic and $\mathrm{V}_{\mathrm{n_k}} \prec \mathrm{V}_{\varepsilon_k}$, than f_{n_k} and f_{n_l} are $\mathrm{V}_{\varepsilon_k}$ homotopic. ■

Proximate subsequence (f_{n_k}) from X to Y over $(\mathrm{V}_{\varepsilon_k})$ is homotopic to proximate sequence (f_n) from X to Y over $(\mathrm{V_n})$. Therefore, from the class of proximate sequences $[(f_n)]$ from X to Y there exist proximate sequence (f_n) over cofinal sequence of finite coverings $(\mathrm{V}_{\varepsilon_n})$, such that each covering $\mathrm{V}_{\varepsilon_n}$ is consisting of ε_n balls.

If in proposition 4, we put $\mathrm{W_n} = \mathrm{V_n} \cup \mathrm{V}_{\varepsilon_1}$, for $n < n_1$ and $\mathrm{W_n} = \mathrm{V}_{\varepsilon_k}$, for $n_k \leq n < n_{k+1} - 1$, than we obtain proposition 4'.

Proposition 4'. Let (f_n) be proximate sequence from X to Y over (V_n). Than there exist a cofinal sequence of finite coverings (W_n), such that starting from some number n_0, coverings W_n consist of ε_n balls.■

If (f_n) from X to Y is proximate sequence over $(\mathrm{W}_{\varepsilon_n})$, and (f'_n) from X to Y is proximate sequence over $(\mathrm{W}'_{\varepsilon_n})$, than there exist cofinal sequence of finite coverings $(\mathrm{V}_{\varepsilon_n})$ such that (f_n) and (f'_n) are proximate sequences over $(\mathrm{V}_{\varepsilon_n})$. Namely elements of $\mathrm{V}_{\varepsilon_n}$ are balls with centers in points which are centers of balls in $\mathrm{W}_{\varepsilon_n}$ and $\mathrm{W}'_{\varepsilon_n}$ and their radius is the maximum of radius of balls in $\mathrm{W}_{\varepsilon_n}$ and $\mathrm{W}'_{\varepsilon_n}$.

To a proximate sequence (f_n) from X to Y, we can associate a proximate net (f_n) from X to Y. Namely, for proximate sequence (f_n) of functions $f_n : X \to Y$ over $(\mathrm{V}_{\varepsilon_n})$, such that starting from some number n_0, $\mathrm{V}_{\varepsilon_n}$ are consisting of ε_n balls, for $n \geq n_0$ every $\mathrm{V}_{\varepsilon_n}$-homotopy which is connecting $\mathrm{V}_{\varepsilon_n}$-continuous functions f_n and f_m, $m > n$, is ε_n- homotopy, which is connecting ε_n-continuous functions f_n and f_m, $m > n$.

Proposition 5. If two proximate sequences (f_n) and (f'_n) from X to Y are homotopic, than the associated proximate nets are homotopic.

Proof. Let two proximate nets (f_n) and (f'_n) from X to Y are homotopic. Analogously as in proposition 4, we can choose, starting from some n_0, cofinal sequence from finite coverings $(\mathrm{V}_{\varepsilon_n})$ which is consisting of open balls of radius ε_n, for which hold for $n \geq n_0$, $F_n(x,0) = f_n(x)$ and $F_n(x,1) = f'_n(x)$, F is $\mathrm{st}\,\mathrm{V}_{\varepsilon_n}$-continuous function and $F_n|_{X \times N_n}$ is $\mathrm{V}_{\varepsilon_n}$ – continuous for some neighborhood $N_n = [0, \varepsilon_n) \cup (1 - \varepsilon_n, 1]$ of $\partial \mathrm{I}$. Than F_n is $\mathrm{V}_{\varepsilon_n}$-continuous function. According to proposition 2 and proposition 3, F_n is $3\varepsilon_n$-continuous function which is connecting functions f_n and f'_n. Therefore proximate nets (f_n) and (f'_n) from X to Y are homotopic.■

Also if (f_n) from X to Y is a proximate net, than we can obtain a proximate sequence (f_n) from X to Y, and if two proximate nets (f_n) and (f'_n) from X to Y are homotopic, than the proximate sequences obtained from them are homotopic.

Using these results in [1] is established functor from the category InSh to category NH.

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The Maximal Subsemigroups of the Semigroup of All Partial Order-Preserving Isometries

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Abstract: *Let I_n be the symmetric inverse semigroup on an n-element set X_n and let ODP_n be its subsemigroup of all partial order-preserving isometries on X_n. In this paper we characterize the maximal subsemigroups of the semigroup ODP_n.*

Keywords: *finite transformation semigroup, partial order-preserving isometries, maximal subsemigroups.*

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Let $X_n = \{1,2,\ldots,n\}$ be an n-element set of the first $n \geq 2$ natural numbers under the natural ordering and let I_n be the symmetric inverse semigroup on X_n, i.e. the partial one-to-one transformation semigroup on X_n under composition of mappings. The importance of I_n to inverse semigroup theory may be likened to that of the symmetric group S_n to group theory. Every finite inverse semigroup S is embeddable in I_n, the analogue of Cayley's theorem for finite groups, and to the regular representation of finite semigroups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of I_n, see for example [1],[4],[6],[13].

We begin by recalling some notations and definitions that will be used in the paper. For standard concepts in semigroup and symmetric inverse semigroup theory, see for example [7] and [10]. We denote by $\mathrm{d}om\,\alpha$ and $\mathrm{i}m\,\alpha$ the domain and the image (range) of $\alpha \in I_n$, respectively. The natural number $\mathrm{rank}\,\alpha := |\mathrm{im}\,\alpha|$ is called the rank of α. We denote by $F(\alpha) := \{x \in X_n \mid x\alpha = x\}$ the set of all fixed points of α. For a subset $A \subseteq I_n$, we denote by $\langle A \rangle$ the subsemigroup of I_n generated by A. A transformation $\alpha \in I_n$ is called order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all x, y in the domain of α. A transformation $\alpha \in I_n$ is called an isometry (or distance-preserving) if $|x - y| = |x\alpha - y\alpha|$ for all x, y in the domain of α. The set of all partial order-preserving isometries on X_n forms a semigroup, denoted by ODP_n.

Proposition 1 ([9]) *The semigroup ODP_n is an inverse subsemigroup of I_n.*

Some semigroups of partial isometries have been studied in [2], [8], [9], [14]. In this paper, we characterize the maximal subsemigroups of the semigroup ODP_n. The maximal subsemigroups of the transformation semigroups have been studied by many authors. For instance, Nichols ([11]) as well as Reilly ([12]) have studied the maximal inverse subsemigroups of the full transformation semigroup T_n. Yang Xiuliang ([15]) has determined the maximal inverse subsemigroups of the finite symmetric inverse semigroup I_n. A complete classification of all maximal subsemigroups of the semigroup O_n of all order-preserving full transformations was obtained by Yang Xiuliang in [16]. Ganyushkin and Mazorchuk ([5]) gave a description of the maximal subsemigroups of the semigroup POI_n of all order-preserving partial injections. Dimitrova and Koppitz ([3]) characterized the maximal subsemigroups of the semigroup $PODI_n$ of all order-preserving or order-reversing partial injections.

For $0 \leq k \leq n$, let

$$I(n,k) := \{\alpha \in ODP_n \mid \mathrm{rank}\,\alpha \leq k\}$$

be the two-sided ideals of ODP_n, consisting of all partial order-preserving isometries with rank not more than k. Every principal factor on ODP_n is a Rees quotient $I(n,k)/I(n,k-1)$ ($1 \leq k \leq n$). The non-zero elements of $I(n,k)/I(n,k-1)$ may be thought of as the elements of ODP_n with rank k precisely. The product of two elements of $I(n,k)/I(n,k-1)$ is 0 whenever their product in ODP_n is with rank strictly less than k.

Let

$$J_k := \{\alpha \in ODP_n \mid \mathrm{r}ank\ \alpha = k\},$$

for $0 \le k \le n$. It is obvious that $I(n,k)$ is the union of the sets $J_0, \dots, J_k$. We will pay attention to the sets J_n and J_{n-1}. The set J_n contains exactly one element, namely the identity mapping, which we denote by ε.

Proposition 2 ([9]) *Let* $\alpha \in ODP_n$ *and* $|F(\alpha)| \ge 1$. *Then* α *is an idempotent.*

It is easy to verify that the elements in J_{n-1} are exactly the following:

$$\varepsilon_i := \begin{pmatrix} 1 & 2 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}, \ for\ i = 1, 2, \dots, n,$$

$$\alpha_1 := \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \ and\ \alpha_n := \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix}.$$

Thus, the set J_{n-1} contains exactly $n+2$ elements and only α_1 and α_n are not idempotents.

Definition 1 *We say that an element* $\alpha \in ODP_n$ *is undecomposable in* ODP_n *if there are not elements* $\beta, \gamma \in ODP_n \setminus \{\alpha\}$ *such that* $\alpha = \beta\gamma$.

Obviously, $\varepsilon \in ODP_n$ is undecomposable in ODP_n. Moreover, it is clear that if $\alpha \in J_{n-1}$ and $\alpha = \beta\gamma$ then $\beta, \gamma \in J_{n-1}$ and $\mathrm{d}om\ \beta = \mathrm{d}om\ \alpha$ as well as $\mathrm{i}m\ \gamma = \mathrm{i}m\ \alpha$. It is easy to check that

$$\alpha_1\alpha_n = \varepsilon_1, \quad \alpha_n\alpha_1 = \varepsilon_n, \quad \varepsilon_1\alpha_1 = \alpha_1\varepsilon_n = \alpha_1, \quad \varepsilon_n\alpha_n = \alpha_n\varepsilon_1 = \alpha_n.$$

All other products of the elements in J_{n-1} are 0. Therefore, we obtain

Proposition 3 *The elements of the set* $J_{n-1} \setminus \{\varepsilon_1, \varepsilon_n\}$ *are undecomposable in* ODP_n.

Corollary 1 $J_{n-1} \subseteq \langle \varepsilon_2, \varepsilon_3, \dots, \varepsilon_{n-1}, \alpha_1, \alpha_n \rangle$.

Proposition 4 *Let* $0 \le k \le n-2$. *Then* $J_k \subseteq \langle J_{k+1} \rangle$.

Proof: Let $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix} \in J_k$. We will show that there exist $\beta, \gamma \in J_{k+1}$ such that $\alpha = \beta\gamma$. We consider three cases for the elements in $\mathrm{d}om\,\alpha = \{a_1, a_2, \ldots, a_k\}$ and $\mathrm{i}m\,\alpha = \{b_1, b_2, \ldots, b_k\}$.

Case 1. There exists $i \in \{1, \ldots, k-1\}$ such that $|\, a_{i+1} - a_i \,| = |\, b_{i+1} - b_i \,| \geq 3$. Then

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_i & a_i + 1 & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & b_i + 1 & b_{i+1} & \cdots & b_k \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_i & b_i + 2 & b_{i+1} & \cdots & b_k \\ b_1 & \cdots & b_i & b_i + 2 & b_{i+1} & \cdots & b_k \end{pmatrix}.$$

Case 2. Let $|\, a_{l+1} - a_l \,| = |\, b_{l+1} - b_l \,| \leq 2$ for all $l \in \{1, \ldots, k-1\}$. Then we consider two subcases:

2.1. There exist at least two indexes $i, j \in \{1, \ldots, k-1\}$ such that $|\, a_{i+1} - a_i \,| = |\, b_{i+1} - b_i \,| = 2$ and $|\, a_{j+1} - a_j \,| = |\, b_{j+1} - b_j \,| = 2$. Then

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_i & a_i + 1 & a_{i+1} & \cdots & a_j & a_{j+1} & \cdots & a_k \\ b_1 & \cdots & b_i & b_i + 1 & b_{i+1} & \cdots & b_j & b_{j+1} & \cdots & b_k \end{pmatrix}$$

$$\begin{pmatrix} b_1 & \cdots & b_i & b_{i+1} & \cdots & b_j & b_j + 1 & b_{j+1} & \cdots & b_k \\ b_1 & \cdots & b_i & b_{i+1} & \cdots & b_j & b_j + 1 & b_{j+1} & \cdots & b_k \end{pmatrix}.$$

2.2. There exists $i \in \{1, \ldots, k-1\}$ such that $|\, a_{i+1} - a_i \,| = |\, b_{i+1} - b_i \,| = 2$ and $|\, a_{j+1} - a_j \,| = |\, b_{j+1} - b_j \,| = 1$ for all $j \neq i$. Then we have $b_1 > 1$ or $b_k < n$, since $k \leq n - 2$. If $b_1 > 1$ then

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_i & a_i + 1 & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & b_i + 1 & b_{i+1} & \cdots & b_k \end{pmatrix} \begin{pmatrix} b_1 - 1 & b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k \\ b_1 - 1 & b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k \end{pmatrix}.$$

If $b_k < n$ then

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_i & a_i + 1 & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & b_i + 1 & b_{i+1} & \cdots & b_k \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k & b_k + 1 \\ b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k & b_k + 1 \end{pmatrix}.$$

Case 3. Let $|\, a_{i+1} - a_i \,| = |\, b_{i+1} - b_i \,| = 1$ for all $i \in \{1, \ldots, k-1\}$. Then we consider the following subcases:

3.1. If $a_k < n$ and $b_k < n - 1$ then

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k & a_k + 1 \\ b_1 & \cdots & b_k & b_k + 1 \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_k & b_k + 2 \\ b_1 & \cdots & b_k & b_k + 2 \end{pmatrix}.$$

3.2. If $a_k < n$ and $b_k = n-1$ then $b_1 > 1$ (since $k \le n-2$) and

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k & a_k+1 \\ b_1 & \cdots & b_k & b_k+1 \end{pmatrix} \begin{pmatrix} b_1-1 & b_1 & \cdots & b_k \\ b_1-1 & b_1 & \cdots & b_k \end{pmatrix}.$$

3.3. If $a_k < n$ and $b_k = n$ then $b_1 > 2$ and

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & a_k+1 \\ b_1-1 & b_1 & \cdots & b_{k-1} & b_k \end{pmatrix} \begin{pmatrix} b_1-2 & b_1-1 & b_1 & \cdots & b_{k-1} \\ b_1-1 & b_1 & b_2 & \cdots & b_k \end{pmatrix}.$$

3.4. If $a_k = n$ and $b_k < n-1$ then $a_1 > 2$ and

$$\alpha = \begin{pmatrix} a_1-1 & a_1 & \cdots & a_{k-1} & a_k \\ b_1 & b_2 & \cdots & b_k & b_k+1 \end{pmatrix} \begin{pmatrix} b_2 & b_3 & \cdots & b_k & b_k+1 & b_k+2 \\ b_1 & b_2 & \cdots & b_{k-1} & b_k & b_k+1 \end{pmatrix}.$$

3.5. If $a_k = n$ and $b_k = n-1$ then $a_1 > 2$, $b_1 > 1$ and

$$\alpha = \begin{pmatrix} a_1-1 & a_1 & a_2 & \cdots & a_k \\ b_1-1 & b_1 & b_2 & \cdots & b_k \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \cdots & b_k & b_k+1 \\ b_1 & b_2 & \cdots & b_k & b_k+1 \end{pmatrix}.$$

3.6. If $a_k = n$ and $b_k = n$ then $a_1 > 2$, $b_1 > 2$ and

$$\alpha = \begin{pmatrix} a_1-1 & a_1 & \cdots & a_k \\ b_1-1 & b_1 & \cdots & b_k \end{pmatrix} \begin{pmatrix} b_1-2 & b_1 & \cdots & b_k \\ b_1-2 & b_1 & \cdots & b_k \end{pmatrix}.$$

Corollary 2 *Let* $0 \le k \le n-1$. *Then* $I(n,k) = \langle J_k \rangle$ *and* $ODP_n = \langle J_{n-1} \cup J_n \rangle$ *.*

Now, we present the characterization of the maximal subsemigroups of the semigroup ODP_n.

Lemma 1 *Every maximal subsemigroup of* ODP_n *contains the ideal* $I(n,n-2)$.

Proof: Let S be a maximal subsemigroup of ODP_n. If $J_{n-1} \subseteq S$ then $I(n,n-2) \subseteq I(n,n-1) = \langle J_{n-1} \rangle \subseteq S$ by Corollary 2. If $J_{n-1} \not\subseteq S$ then $J_{n-1} \not\subseteq \langle S \cup I(n,n-2) \rangle$ since $I(n,n-2)$ is an ideal with $I(n,n-2) \cap J_{n-1} = \varnothing$. This implies $I(n,n-2) \subseteq S$ by the maximality of S.

Theorem 1 *A subsemigroup* S *of* ODP_n *is maximal if and only if it belongs to one of the following three types:*

$S_{\varepsilon} := ODP_n \setminus \{\varepsilon\}$;
$S_{\varepsilon_i} := ODP_n \setminus \{\varepsilon_i\}$, for $i = 2,3,\ldots,n-1$;
$S_{\alpha_i} := ODP_n \setminus \{\alpha_i\}$, for $i = 1, n$.

Proof: It is clear that S_{ε} is a maximal subsemigroup of ODP_n, since $ODP_n \setminus \{\varepsilon\} = I(n,n-1)$ and $I(n,n-1) \cup \{\varepsilon\} = ODP_n$.

From Proposition 3, we have that α_1, α_n and ε_i, for $i = 2,3,\ldots,n-1$ can not be generated by the elements in J_{n-1}. Therefore, S_{ε_i} as well as S_{α_1} and S_{α_n} are subsemigroups of ODP_n. It is clear that they are maximal since $S_{\varepsilon_i} \cup \{\varepsilon_i\} = ODP_n$ as well as $S_{\alpha_1} \cup \{\alpha_1\} = ODP_n$ and $S_{\alpha_n} \cup \{\alpha_n\} = ODP_n$.

For the converse part, let S be a maximal subsemigroup of ODP_n. Then $S = I(n,n-2) \cup T$, where $T \subset (J_{n-1} \cup J_n)$ (see Lemma 1). If $J_n \not\subseteq T$ then $S \subseteq S_{\varepsilon}$ since $J_n = \{\varepsilon\}$ and thus $S = S_{\varepsilon}$ by the maximality of S. Let $J_n \subseteq T$. Then $J_{n-1} \not\subseteq T$. Since $J_{n-1} \subseteq \langle \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{n-1}, \alpha_1, \alpha_n \rangle$, by Corollary 1, the set T is contained in $J_{n-1} \setminus \{\beta\}$ for some $\beta \in \{\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{n-1}, \alpha_1, \alpha_n\}$. Therefore, $S = S_{\varepsilon_i}$ for $i = 2,3,\ldots,n-1$ or $S = S_{\alpha_j}$ for $j = 1, n$ by the maximality of S.

Corollary 3 *The semigroup ODP_n contains exactly $n+1$ maximal subsemigroups.*

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Systems of Difference Equations as a Model for the Lorenz System

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1. INTRODUCTION

The use of power series is one of the oldest methods for examining differential equations. In one way it is used as a numerical method for solving differential equations, and on the other it is used also for theoretical results. In the literature there are numerous papers concerned with such a use of power series, like the papers [1], [2] and [3].

In [4] we have used power series combined with difference equations to find a local approximations to the solution of the Lorenz system of differential equations:

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= x(r-z)-y \\ \dot{z} &= xy-bz \end{aligned} \tag{1.1}$$

with parameters *σ, r, b*. For initial values $a_0 = x(0), b_0 = y(0), c_0 = z(0)$, assuming the solutions of the system are expanded as Maclaurin series,

$$\begin{aligned} x(t) &= a_0 + a_1 t + a_2 t^2 / 2! + \ldots + a_n t^n / n! + \ldots \\ y(t) &= b_0 + b_1 t + b_2 t^2 / 2! + \ldots + b_n t^n / n! + \ldots \\ z(t) &= c_0 + c_1 t + c_2 t^2 / 2! + \ldots + c_n t^n / n! + \ldots \end{aligned} \tag{1.2}$$

using several systems of difference equations we produced formulas for approximations of the coefficients in the Maclaurin series.

Using these formulas for coefficients in the Maclaurin polynomials, we obtained functions that approximate the solutions of the Lorenz system.

In this paper we consider systems of difference equations, whose solutions can be found in the same way as the formulas produced in [4].

2. SYSTEMS OF DIFFERENCE EQUATIONS

For the Lorenz system (1.1) with the parameters *σ, r, b*, and initial values $a_0 = x(0), b_0 = y(0), c_0 = z(0)$, for any positive integers $K_a, K_b, K_c > 4$ we consider the following systems of difference equations, for a_n, b_n, c_n:

(2.1)
$$\begin{aligned} a_n &= -Aa_{n-1} + Ba_{n-2} - Ca_{n-3} + Da_{n-4} \\ &\quad + (-1)^{n-5}\left(H_n - AH_{n-1} - BH_{n-2} - CH_{n-3} - DH_{n-4}\right),\ n > K_a \end{aligned}$$

(2.2)
$$\begin{aligned} b_n &= -Ab_{n-1} + Bb_{n-2} - Cb_{n-3} + Db_{n-4} \\ &\quad + (-1)^{n-5}\left(W_n - AW_{n-1} - BW_{n-2} - \overline{C}W_{n-3} - DW_{n-4}\right), \quad n > K_b \end{aligned}$$

(2.3)
$$\begin{aligned} c_n &= -Ac_{n-1} + Bc_{n-2} - Cc_{n-3} + Db_{n-4} \\ &\quad + (-1)^{n-5}\left(-b^{n-4}c_0\right)\left(b^4 - Ab^3 - Bb^2 - Cb - D\right), \quad n > K_c \end{aligned}$$

where:

(2.4)
$$\begin{aligned} H_n &= (\sigma + r)^{n-1}(b_0 - a_0) + \sum_{m=1}^{\left[\frac{n-1}{2}\right]} \sum_{j=m+1}^{n-m} \binom{n-j}{m}\binom{j-1}{m-1} \sigma^{n-j-1}\left(r^j - r^m\right)a_0 \\ &\quad - \sum_{m=1}^{\left[\frac{n}{2}\right]} \sum_{j=m}^{n-m} \binom{n-1-j}{m-1}\binom{j}{m-1} \sigma^{n-j-1}\left(r^j - r^{m-1}\right)b_0 \end{aligned};$$

(2.5)
$$\begin{aligned} W_n &= \left\{\left[\sigma + (r - c_0)\right]^{n-1}(a_0 - b_0)(r - c_0) + \left[(r - c_0)^n - 1\right]b_0\right\} + \\ &\quad + \left\{ -\sum_{m=1}^{\left[\frac{n}{2}\right]} \sum_{j=m+1}^{n-m+1} \binom{n-j}{m-1}\binom{j-1}{m-1} \sigma^{n-j}\left((r - c_0)^j - (r - c_0)^m\right)a_0 \right\} \\ &\quad + \left\{ \sum_{m=2}^{\left[\frac{n+1}{2}\right]} \sum_{j=m}^{n-m+1} \binom{n-1-j}{m-2}\binom{j}{m-1} \sigma^{n-j}\left((r - c_0)^j - (r - c_0)^{m-1}\right)b_0 \right\} \end{aligned};$$

$A = 1 + \sigma + b,\ B = \sigma r - a_0^2 - \sigma c_0,\ C = \sigma a_0 b_0,\ \overline{C} = \sigma a_0 b_0 - \sigma b c_0,\ D = -\sigma^2 b_0^2$;

and a_p, b_q, c_r for $p \le K_a, q \le K_b, r \le K_c$ are calculated from the Lorenz system (1.1) as the corresponding derivatives: $a_p = \overset{(p)}{x}(0), b_q = \overset{(q)}{y}(0), c_r = \overset{(r)}{z}(0)$.

Below we give the formulas for $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4$:

$$a_1 = \sigma(b_0 - a_0)$$

$$a_2 = \sigma a_0(r - c_0) - \sigma b_0 - \sigma^2 b_0 + \sigma^2 a_0$$

$$a_3 = -\sigma^3 a_0 + \sigma^3 b_0 + \sigma^2 b_0 + \sigma b_0 + \sigma b a_0 c_0 - \sigma a_0{}^2 b_0 + \sigma^2 b_0 (r - c_0) \\ - \sigma a_0 (r - c_0) - 2\sigma^2 a_0 (r - c_0)$$

$$a_4 = \sigma^4 a_0 - \sigma b_0 - \sigma^2 b_0 - \sigma^3 b_0 - \sigma^4 b_0 - \sigma b a_0 c_0 - \sigma b^2 a_0 c_0 - 3\sigma^2 b a_0 c_0 \\ + 2\sigma^2 b b_0 c_0 - 3\sigma^2 a_0 b_0{}^2 + \sigma b a_0{}^2 b_0 + 2\sigma a_0{}^2 b_0 + 4\sigma^2 a_0{}^2 b_0 + \sigma a_0 (r - c_0) \\ - \sigma a_0{}^3 (r - c_0) + 2\sigma^2 a_0 (r - c_0) + 3\sigma^3 a_0 (r - c_0) - 2\sigma^3 b_0 (r - c_0) \\ - 2\sigma^2 b_0 (r - c_0) + \sigma^2 a_0 (r - c_0)^2$$

$$b_1 = a_0 (r - c_0) - b_0$$

$$b_2 = -\sigma a_0 (r - c_0) - a_0 (r - c_0) + \sigma b_0 (r - c_0) + b_0 - a_0{}^2 b_0 + b a_0 c_0$$

$$b_3 = \sigma^2 a_0 (r - c_0) + \sigma a_0 (r - c_0)^2 + \sigma a_0 (r - c_0) + a_0 (r - c_0) - \sigma^2 b_0 (r - c_0) \\ - 2\sigma b_0 (r - c_0) + + 2a_0{}^2 b_0 + 3\sigma a_0{}^2 b_0 + b a_0{}^2 b_0 - a_0{}^3 (r - c_0) - 3\sigma b_0{}^2 a_0 \\ - 2\sigma b a_0 c_0 - b a_0 c_0 + 2\sigma b b_0 c_0 - b_0 - b^2 a_0 c_0$$

$$b_4 = -2\sigma^2 a_0 (r - c_0)^2 - 2\sigma a_0 (r - c_0)^2 + \sigma^2 b_0 (r - c_0)^2 - 11\sigma a_0^2 b_0 (r - c_0) \\ + 4\sigma b a_0 c_0 (r - c_0) + 3\sigma b_0 (r - c_0) + 2\sigma^2 b_0 (r - c_0) + \sigma^3 b_0 (r - c_0) \\ - \sigma^3 a_0 (r - c_0) - \sigma^2 a_0 (r - c_0) - \sigma a_0 (r - c_0) + 6\sigma a_0{}^3 (r - c_0) \\ + 2a_0{}^3 (r - c_0) - a_0 (r - c_0) + b a_0{}^3 (r - c_0) - 8\sigma a_0{}^2 b_0 - 3a_0{}^2 b_0 \\ - 2b a_0{}^2 b_0 + 12\sigma a_0 b_0{}^2 - 5\sigma b b_0 c_0 + 2\sigma b a_0 c_0 + b^2 a_0 c_0 + b_0 + 10\sigma^2 a_0 b_0{}^2 \\ + 4\sigma b a_0 b_0{}^2 - b^2 \sigma^2 a_0{}^2 b_0 - 7\sigma^2 a_0{}^2 b_0 + a_0{}^4 b_0 + b^3 a_0 c_0 - b a_0^3 c_0 \\ - 4\sigma b a_0^2 b_0 - 3\sigma^2 b_0^3 - 3\sigma b^2 b_0 c_0 - 3\sigma^2 b b_0 c_0 + + 3\sigma^2 b a_0 c_0 + 3\sigma b^2 a_0 c_0$$

$$c_1 = a_0 b_0 - b c_0$$

$$c_2 = -a_0 b_0 - \sigma a_0 b_0 - b a_0 b_0 + \sigma b_0^2 + b^2 c_0 + a_0^2 (r - c_0)$$

$$c_3 = 4\sigma a_0 b_0 (r - c_0) - 3\sigma a_0^2 (r - c_0) - a_0^2 (r - c_0) - b a_0^2 (r - c_0) \\ - 3\sigma b_0^2 - \sigma^2 b_0^2 - \sigma b b_0^2 + a_0 b_0 + 2\sigma a_0 b_0 + b a_0 b_0 + \sigma^2 a_0 b_0 + \\ + \sigma b a_0 b_0 + b^2 a_0 b_0 - a_0^3 b_0 - b^3 c_0 + b a_0^2 c_0$$

$$
\begin{aligned}
c_4 = & -a_0b_0 + 2a_0^3b_0 - a_0^2c_0 + a_0^4c_0 + a_0^2r - a_0^4r - ba_0b_0 + 2ba_0^3b_0 \\
& - 2ba_0^2c_0 + bra_0^2 - b^2a_0b_0 - 3b^2a_0^2c_0 + b^2ra_0^2 - b^3a_0b_0 - \\
& - 3\sigma a_0b_0 + 6\sigma a_0^3b_0 + 7\sigma b_0{}^2 - 7\sigma a_0{}^2b_0{}^2 - 4\sigma a_0{}^2c_0 + \\
& + 12\sigma a_0b_0c_0 + 4\sigma a_0{}^2c_0{}^2 + 4\sigma ra_0{}^2 - 12\sigma ra_0b_0 - 8\sigma ra_0{}^2c_0 \\
& + 4\sigma r^2a_0{}^2 - 2\sigma ba_0b_0 + 3\sigma bb_0^2 - 8\sigma ba_0{}^2c_0 + 10\sigma ba_0b_0c_0 + \\
& + 3\sigma bra_0{}^2 - 4\sigma b\,ra_0b_0 - \sigma b^2a_0b_0 + \sigma b^2b_0{}^2 - 3\sigma^2a_0b_0 + 4\sigma^2b_0{}^2 \\
& - 7\sigma^2a_0{}^2c_0 + 12\sigma^2a_0b_0c_0 - 4\sigma^2b_0{}^2c_0 + 7\sigma^2ra_0{}^2 - 12\sigma^2ra_0b_0 + \\
& 4\sigma^2rb_0{}^2 - \sigma^2ba_0b_0 + \sigma^2bb_0{}^2 - \sigma^3a_0b_0 + \sigma^3b_0{}^2 + b^4c_0
\end{aligned}
$$

As mentioned above, the solutions of the systems (2.1), (2.2), and (2.3) can be found in the same way as the formulas produced in [4].

3. FUNCTIONS $x_T(t), y_T(t), z_T(t)$

If we take the solutions of the systems (2.1), (2.2) and (2.3) as coefficients in the power series (1.2), we obtain three power series. At this moment the question of what conditions would imply the convergence of these power series is open.

For given *σ, r, b,* a_0, b_0, c_0, positive integer *m* and positive integers $K_a, K_b, K_c > 4$, we consider the polynomials:

$$
\begin{aligned}
& P_m(a_0,b_0,c_0)(t) = a_0 + a_1t + a_2t^2/2! + \ldots + a_nt^m/m! \\
(3.1) \qquad & Q_m(a_0,b_0,c_0)(t) = b_0 + b_1t + b_2t^2/2! + \ldots + b_nt^m/m! \\
& R_m(a_0,b_0,c_0)(t)(t) = c_0 + c_1t + c_2t^2/2! + \ldots + c_nt^m/m!
\end{aligned}
$$

where a_t, b_t, c_t are the solutions of the systems (2.1), (2.2) and (2.3).

Let *T* be a positive real number and m be a positive integer. For given *σ, r, b,* a_0, b_0, c_0, *m* and $K_a, K_b, K_c > 4$, we define functions $x_T(t), y_T(t), z_T(t)$ for $t \in [0, \infty)$ as follows. For $t \in [0, T]$,

$$x_T(t) = P_m(a_0,b_0,c_0)(t),\ y_T(t) = Q_m(a_0,b_0,c_0)(t),\ z_T(t) = R_m(a_0,b_0,c_0)(t).$$

Next, continue by induction. Assume that $x_T(t), y_T(t), z_T(t)$ are defined for $[0, k\cdot T]$. Extend them on $[0, (k+1)\cdot T]$ such that for $t \in [k\cdot T, (k+1)\cdot T]$:

$$x_T(t) = P_m(u_0,v_0,w_0)(t - k\cdot T),\ \ y_T(t) = Q_m(u_0,v_0,w_0)(t - k\cdot T) \text{ and}$$

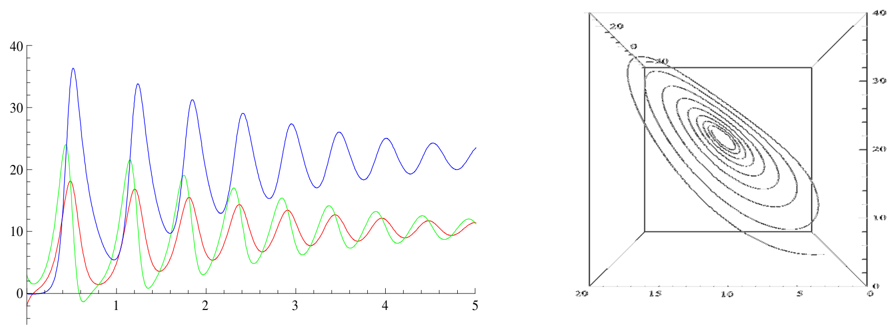
$$z_T(t) = R_m(u_0,v_0,w_0)(t - k\cdot T),$$

where $u_0 = x_T(k \cdot T), v_0 = y_T(k \cdot T), \;\; w_0 = z_T(k \cdot T)$.

At this moment we do not have the answer to the question of how good approximations, modulo T and m, are the functions $x_T(t), y_T(t), z_T(t)$ for the solutions of the Lorenz system (1.1).

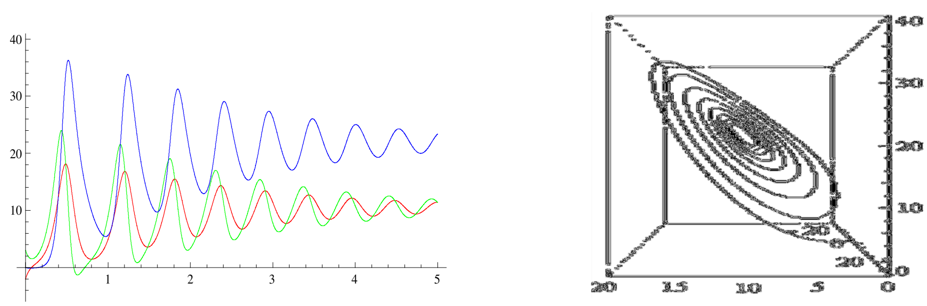
In examples, by computer calculations, for small values of *T*, we obtain that the functions $x_T(t), y_T(t), z_T(t)$ are good approximations for the solutions of the system (1.1). We used the program Mathematica and compared the solutions obtained by the program Mathematica with the functions $x_T(t), y_T(t), z_T(t)$.

Example 3.1. Parameters: $\sigma = 10, r = 23, b = 5$; *T*=0,05; *m*=20; initial values $a_0 = -2, b_0 = 3, c_0 = 0$; and the time interval [0,5].



a) Graphics of the solutions **b)** Space curve solution

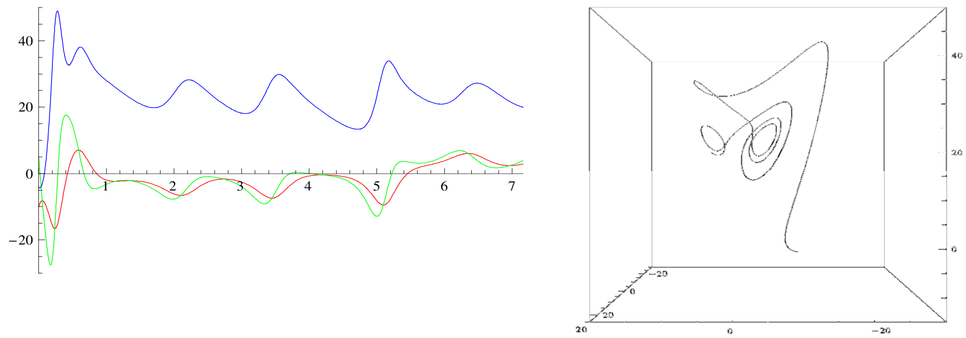
Fig.1: Results obtained by the program Mathematica



a) Graphics of the functions b) Space curve

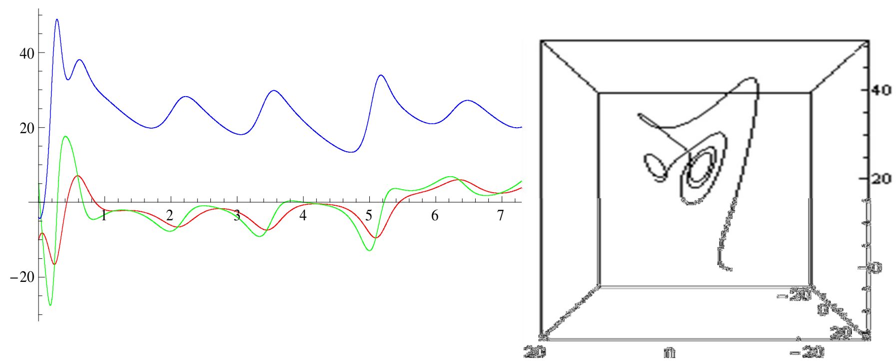
Fig.2: Results obtained by computing the functions $x_T(t), y_T(t), z_T(t)$

Example 3.2: Parameters: $\sigma = 5, r = 25, b = 0.8$; *T*=0,05; *m*=20; initial values $a_0 = -10, b_0 = 5, c_0 = -4$; and the time interval [0,7.35].



a) Graphics of the solutions b) The space curve

Fig.3: Results obtained by the program Mathematica



a) Graphics of the functions b) The space curve

Fig.4: Results obtained by computing the functions $x_T(t), y_T(t), z_T(t)$

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Structure Of a Fuzzy Gamma Module

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Let Γ be an abelian group and R be a Γ- ring. Let M be a R_Γ-module. In this regards we introduce and study fuzzy R_Γ-modules and investigate some basic properties of them.

Keywords: *gamma ring, gamma module, gamma submodule, fuzzy gamma ring, fuzzy gamma modules.*

1 Introduction

The notion of gamma ring was introduced by N. Nobusawa [3]. The authors in [1] introduced and studied modules over gamma rings. The concepts of fuzzy subset was introduced by Zadeh [5]. This concepts was applied to the theory of module by Negoita and Ralescu. In this paper we follow [1] and study fuzzy gamma modules.

2 Preliminaries

Let R and Γ be abelian groups. We say that R is a *gamma ring* if there exists a mapping

$$\cdot : R\times\Gamma\times R \to R$$
$$(r,\gamma,r') \mapsto r\gamma r'$$

such that for every $a,b,c\in R$ and $\alpha,\beta\in\Gamma$, the following conditions are satisfied:

(GR_1) (i) $(a+b)\alpha c = a\alpha c + b\alpha c$;

(ii) $a(\alpha+\beta)c = a\alpha c + a\beta c$;

(iii) $a\alpha(b+c) = a\alpha b + a\alpha c$;

(GR_2) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Definition 2.1. ([2]) Let R be a Γ-ring. A $(left)$ *gamma module* over R is an additive abelian group M together with a mapping

$.: R\times\Gamma\times M \to M$, such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $r, r_1, r_2 \in R$ the following conditions are satisfied:

(GM_1) $r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$;

(GM_2) $(r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m$;

(GM_3) $r(\gamma_1 + \gamma_2)m = r\gamma_1 m + r\gamma_2 m$;

(GM_4) $r_1\gamma_1(r_2\gamma_2 m) = (r_1\gamma_1 r_2)\gamma_2 m.$

Definition 2.2. Let M be an R_Γ-module. A nonempty subset N of M is said to be a *left gamma submodule* of M if N is a subgroup of M and $R\Gamma N \subseteq N$, where $R\Gamma N = \{\sum_{i=1}^{n} r_i\gamma_i n_i \mid r_i \in R, \gamma_i \in \Gamma, n_i \in N, n \in N\}$.

3 Fuzzy Gamma Modules

Definition 3.1. A fuzzy subset μ_R of Γ-ring R is called a fuzzy gamma ring if for all $x, y \in R, \gamma \in \Gamma,$ we have

(1) $\mu_R(x - y) \geq \mu_R(x) \wedge \mu_R(y)$;

(2) $\mu_R(x\gamma y) \geq \mu_R(x) \wedge \mu_R(y)$.

Definition 3.2. A fuzzy subset μ_M of Γ-module M is called a fuzzy Γ-module over a fuzzy gamma ring μ_R if for all $x, y \in M$, $r \in R$, we have

(1) $\mu_M(x - y) \geq \mu_M(x) \wedge \mu_M(y)$;

(2) $\mu_M(r\gamma x) \geq \mu_R(r) \wedge \mu_M(x).$

In brief μ_M is a fuzzy (μ_R, Γ)-module.

Definition 3.3. Let M ba a gamma module, and μ_M, ν_M are fuzzy subset of M, $r \in R, \gamma \in \Gamma$. We define fuzzy subsets $\mu_M \cap \nu_M, \mu_M + \nu_M, r\gamma\mu_M, -\mu_M$ of M in the following ways for $x \in M$,

$$(\mu_M \cap \nu_M)(x) = \mu_M(x) \wedge \nu_M(x)$$

$$(\mu_M + \nu_M)(x) = \vee_{x_1 + x_2 = x} [\mu_M(x_1) \wedge \nu_M(x_2)]$$

$$(r\gamma\mu_M)(x) = \vee_{r\gamma x_1 = x}\, \mu_M(x_1)$$

$$(-\mu_M)(x) = \mu_M(-x)$$

Definition 3.4. Define the direct sum, and the composition, and product respectively as follows: for all $x \in M$,

$$(\mu_M \oplus \nu_M)(x) = sup_{x=u+v}\{min[\mu_M(u), \nu_M(v)]\};$$
$$(\mu_R \circ \nu_M)(x) = sup\{min[\mu_R(r), \nu_M(u)] \mid r \in R, u \in m, x = r\gamma u, \gamma \in \Gamma\};$$

$$(\mu_R \Gamma \nu_M)(x) = sup\{inf_{i=1}^{n}\{\mu_R(r_i) \wedge \nu_M(x_i)\} \mid r_i \in R, x_i \in M, \gamma_i \in \Gamma, n \in \mathbf{N}, x = \Sigma_{i=1}^{n} r_i \gamma_i x_i\}.$$

We denote by $F\Gamma(M)$ the set of all fuzzy Γ-modules of M over fuzzy Γ-ring μ_R, and $F\Gamma(R)$ the set of all fuzzy Γ-rings of gamma ring R.

Definition 3.5. For all $a \in R$, we define set $J_a^{'} = \{\beta \mid a \in (\mu_R)_\beta\}$, and for all $x \in M$, Let $J_x = \{\alpha \mid x \in (\mu_M)_\alpha\}$, then $\mu_M(x) = supJ_x$.

Proposition 3.6. μ_M is a fuzzy μ_R-gamma module iff

$(1) \vee \mu_M(x) = 1$

$(2) J_{-x} \supseteq J_x, J_{x+y} \supseteq J_x \cap J_y, J_{a\gamma x} \supseteq J_a^{'} \cap J_x$, for all $x, y \in M, a \in r, \gamma \in \Gamma$.

Proposition 3.7. Let $\mu \in F\Gamma(R), \nu \in F\Gamma(M)$. Then $\mu \circ \nu \subseteq \mu\Gamma\nu \subseteq \mu \cap \nu$.

Proposition 3.8. Let $\mu, \nu \in F\Gamma(M)$. Then $\mu \oplus \nu \in F\Gamma(M)$.

dent **Proposition 3.9.** Let $\mu, \nu \in F\Gamma(M)$ and $\xi \in F\Gamma(R)$. Then

(1) $\xi\Gamma\mu \subseteq \nu$ if and only if $\xi \circ \mu \subseteq \nu$.

(2) Let $r_t \in F\Gamma(R), x_s \in F\Gamma(M)$ be fuzzy points and $\gamma \in \Gamma$. Then $r_t \circ x_s = r_t\Gamma x_s = (r\Gamma x)_{t \wedge s}$, such that $r\Gamma x = \{r\gamma x \mid \gamma \in \Gamma\}$.

(3) $\xi(0) = 1$, then $\xi\Gamma\nu \in F\Gamma(M)$;

(4) Let $r_t \in F\Gamma(R)$ be a fuzzy point. Then for all $x \in M$,

$$(r_t \circ \mu)(x) = \begin{cases} t \wedge sup\{\mu(m) \mid x \in M, x = r\gamma m\} & \exists \gamma \in \Gamma \ with \ x = r\gamma m \\ 0 & otherwise \end{cases}$$

Proposition 3.10. Let N be a submodule of Γ-module M . If μ is a fuzzy gamma module, then the fuzzy set $\overline{\mu}$ of M/N defined by

$$\overline{\mu}(a+N) = sup_{x \in N}\ \mu(a+x)$$

is a fuzzy gamma module.

Proposition 3.11. Let M be a gamma module, $\mu \in F\Gamma(M)$. Then M/μ is a gamma module. It is enough we consider the mapping

$$.:(R,\Gamma,M/\mu)\rightarrow M/\mu$$
$$(r,\gamma,(m+\mu))\rightarrow(r.\gamma.m)+\mu$$

Now we define a fuzzy set on M/μ. Let ν be any fuzzy gamma module of M, ν/μ be a fuzzy set of M/μ defined as follows:

$$\nu/\mu : M/\mu \rightarrow [0,1]$$
$$\nu/\mu(m+\mu) = sup_{m+\mu=x+\mu}\,\nu(x),$$

such that $x+\mu = y+\mu \Leftrightarrow \mu(x)=\mu(y)$ for all $m+\mu, x+\mu \in M/\mu$ and $\forall r\in R, \gamma\in\Gamma, M = r\gamma M$. The fuzzy subset ν/μ is a fuzzy gamma module of M/μ.

Proposition 3.12. Let μ be a fuzzy Γ-submodule of M such that R,Γ are commutative groups, and $r\in R, \gamma\in\Gamma$. Then $r\circ\mu$ that is defined $(r\circ\mu)(x)=\mu(r\gamma x)$ is a fuzzy Γ-module.

Proposition 3.13. Let μ be a fuzzy Γ-module of M. Then ν with the mapping $\nu(f)=\wedge_{a\in A}\mu(f(a))$ is a fuzzy Γ-module of M^A.

Proposition 3.14. Let μ be fuzzy Γ-module of M and for every $m,m'\in M$ we have $(m\circ m')(x)=\mu(m+m'-x)$. Then $m\circ m'$ is a fuzzy Γ-module of M, if in definition fuzzy gamma module μ and $m\circ m'$ we changed $\geq$ into equality.

Proposition 3.15. Let μ is a fuzzy Γ-module of M. Then ν that is defined $\nu(f(x))=\wedge_{i=0}^{n}\mu(a_i)$ is a fuzzy Γ-submodule of $M[x]$, for every $f(x)=\sum_{i=0}^{n}a_i x^i$.

Proposition 3.16. Let μ be a fuzzy Γ-module of M. Then μ is a fuzzy Γ-module of $S(M)$ such that $\mu(A)=\wedge_{a\in A}\mu(a)$ for all $A\subseteq M$.

Proposition 3.17. Let μ be a fuzzy Γ-module of M. Then μ is a fuzzy Γ-module of $S'(M)=\{A \mid \forall B\in S(M), A\cap B=\varnothing\}$ such that $(S'(M),\Delta)$ is a Γ-module and for all $A\subseteq M$, $A\subseteq r\gamma A, \mu(A)=\chi_A$, then μ is a fuzzy Γ-submodule of $S'(M)$.

Proposition 3.18. Let μ be a fuzzy Γ-submodule of M. Let $(m \circ m^{'})(t) = \mu(m) \wedge \mu(m^{'})$ if $t = m - m^{'}$. Let for every $\mu \neq 0$ such that $\forall x, y \in M$, $\mu(x - y) = \mu(x) \wedge \mu(y)$, we have

$$(m \otimes \mu)(x) = \vee_{t \in M} ((m \circ t)(x) \wedge \mu(t)).$$

Then $m \otimes \mu$ is a fuzzy Γ-module of M.

Proposition 3.19. Let μ be a fuzzy (R, Γ)-submodule of M. Then $supp(\mu) = \{x \in M \mid \mu(x) > 0\}$ is a $(supp(\mu_R), \Gamma)$-submodule of M.

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EQUATION OF THE FUNCTIONING OF AN AIRCRAFT AND HIS A CRASH FUNCTION

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Abstract: *The article has as a main purpose to research the behavior of the aircraft of transport type. The loss of its stability corresponds to a simple accident. For this aircraft the matrix of stability in the case of some stability stationary solutions has got complex roots. One of its real roots passes through zero and it is responsible for the loss of stability and the occurrence of an accident.*

Keywords: *aircraft, function of the crash*

1. INTRODUCTION

In common case equation of the motion of an aircraft (flying apparatus), is ***a complex, nonlinear first order system***. This nonlinear system has a form:

(1) $$\frac{dx_i}{dt} = F_i(x;c), x \in R^n, c \in R^k,$$

where in (1) it is taken into account n-dimensional x as a system of variables arguments determinant of the state of the aircraft and *k* – dimensional *c* as a system of regulatory parameters.

One of the tasks of the aerodynamics consists in determining of the dependence the solution $x \in R^n$ from the operate parameters $c \in R^k$.

The subject of the theory of catastrophes related with the fling aircraft is determination of the number, type and properties of stability of the solutions of the system (1).

This is why it is needed to find bifurcation set $\Psi_{БМ} \subset R^k$ in which these numbers and kind of solutions is altering. Similarly one must to investigate properties of the phase transitions, i.e., transitions of the system from one stationary state to other.

There exist several approaches for solving this problem, as they are considered in the literature [1-4]. More interesting from them are:

- using of Runge-Kutta algorithm with a suitable software;
- investigation of the behavior at a catastrophe (reaction in a catastrophe) of dynamical system from (1) by preliminary investigation of stationary solutions $dx_i/dt = \dot{x}_i = 0$;
- using the Taylor series concerning force function $F_i(x;c)$ in a vicinity of the corresponding stationary solution of (1)

The present piper is concerning with the last approach assuming that coordinates of the varied state are chosen so that $x = 0$ for $c = 0$. By this Taylor series decomposition of the right part of equation (1), in the chosen state will take the type:

$$\dot{x}_i = F_i + F_i^{\,j} x_i + F_i^{\,jk} x_j x_k + \dots \quad . \tag{2}$$

In equation (2) indexes denote differentiation by corresponding coordinates, for example $F_i^{\,jk} = \partial^2 F_i / \partial x_j \partial x_k$. All derivatives are calculated at the point $(x;c) = (0;c)$. For fixed values of regulatory parameters c the matrix $F_i^{\,j}$ is considered as a function of the linear reaction of the alteration of the changing states, representing the alteration of $\dot{x}_i$ (respectively of x_j). In common case $F_i^{\,jk}$, can be considered as a component of the "**tensor of perceiving**".

Coefficients $F_i(c), F_i^{\,j}(c)$ can be decomposed in Taylor series by regulatory parameters c:

$$F_i(c) = F_i(0) + F_i^{\alpha} c_\alpha + F_i^{\alpha\beta} c_\alpha c_\beta + \dots ,$$

$$F_i^{\,j}(c) = F_i^{\,j}(0) + F_i^{\,j\alpha} c_\alpha + F_i^{\,j\alpha\beta}(c_\alpha c_\beta) + \dots \quad . \tag{3}$$

Coefficients of these decompositions are partial derivatives of the functions F_i, calculated at the point $(x;c) = (0;0) \in R^n \otimes R^k$. Coefficients F_i are considered as a function of a linear reaction of the changing of regulatory parameters of the ***complex nonlinear system first order simple equations***, describing the motion of aircraft.

Coefficients $F_i^j(0,0)$ and $F_i^\alpha(0,0)$ in Taylor series from (3) are calculated in vicinity of stationary state $(0,0)$ and are considered as a function of a linear reaction of alternating of the variables and regulatory parameters. All the rest terms of series turn out nonlinear in alternating state.

2. SOLVING THE PROBLEM

Is considered behavior of aircraft, for which the loose of stability corresponds to one elementary catastrophe. For such an apparatus the stability matrix has complex-conjugate roots, as one of its real roots passes through zero and is responsible for the loose of stability (occurrence of a catastrophe of aircraft). This demands investigation of equation (2). If $x = x^0$ is a stationary solution then linearized matrix of stability in a vicinity of this point becomes of the type

$$M_i^j = F_i^j + 2F_i^{jk} x_k^0 . \tag{4}$$

Before to proceed to describe possible stationary states of the aircraft, it is necessary to choose suitable system of variable conditions. It must be compact, convenient and describing the concrete state of aircraft.

Let us describe the set of the system of coordinates, related with the shell of aircraft. For the representation of the origin of the coordinate system (or system of reading) we put aircraft in the origin of the coordinate system, and the axis of coordinates coincide with the main axis of inertia (Fig. 1).

One introduces the following notations: X, Y, Z are components of the vectors of forces; L, M, N - vector's components of the vector of moment of torque; u, v, w - vector's components of the velocity of the relevant inertial system for reading of motion; p, q, r - vector's components of the angle's velocities of rotation according the main axis; α, β - the attack angle and the angle of sliding; a_l, a_r - angels of ailerons; e_l, e_r - angels of deviation of the rule for attitude; τ - angle of deviation of the rule in direction of the motion of aircraft.

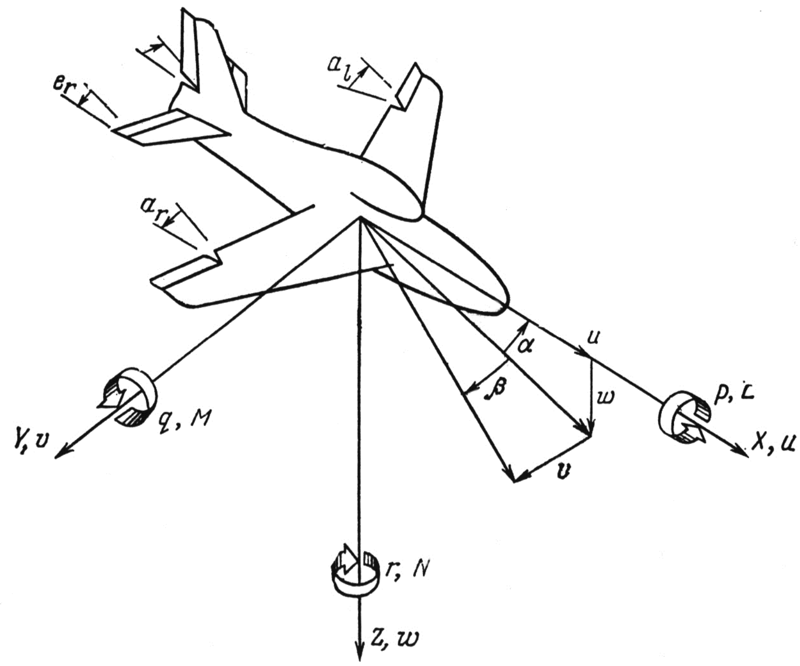

Fig. 1: Coordinate system according construction of aircraft, and coordinate axis of Euler's angles with the axis if inertial system for reading of his motion [1]

Orientation of the axis of motion of aircraft concerning inertial system for reading is determined by the three Euler's angels (ϕ, θ, ψ). The components of the inertia tensor are I_{XX}, I_{YY}, I_{ZZ}. The components of velocity vector ω through main axis are denoted with (p, q, r) and components of force $\vec{T}$ with (L, M, N). The components of the velocity of the center of mass $\vec{V}$, regarding inertial system of reading are denoted by (u, v, w). The components of the force are denoted by (X, Y, Z).

If the velocity of motion of a aircraft (of transport type) is relatively constant it is convenient instead Cartesian coordinate system to use polar coordinate system $|\vec{V}|$, α, β, where α is the angle of attack and β - the angle of sliding of aircraft by its aerodynamic motion in ear fluid. If the angels are small (measured in radians) then the next equalities are valid [1, 2]:

$$\alpha \cong w / |\vec{V}|$$

and

$$\beta \cong v / |\vec{V}| \ .$$

For description of stability of aircraft, it is enough to be used as a “function of state” (***catastrophe function***) – ***CF***, three components of the angle velocity (p,q,r), two “polar” angles and three Euler’s angles (ϕ,θ,ψ).

Hence the space of variable states R^n in ***CF*** is 8 (eight) dimensional and is expressed by the equation

$$F_{\Phi K} = F(p,q,r,\alpha,\beta,\phi,\theta,\psi) = F(x_1,...,x_8) \in R^8 \qquad (5)$$

The orientation of aircraft is determined by the position a_l, a_r of the left end right aileron, position e_l, e_r of the left and right rule for attitude and position τ of the rule for direction. All these angels are measured in the state of flying in right direction and constant height (as a rule they have a small quantity). So calculated angle deviations are expressed as ruling influence of the flight of aircraft. They are defined by the following function of control

$$W(a_l,a_r,e_l,e_r,\tau) = W(c_1,c_2,c_4,c_5) \in R^5 \qquad (6)$$

In equation (6) for convenience the following notations are introduced

$$a = 0{,}5(a_l + a_r), \ \delta a = a_l - a_r, \qquad (7)$$

$$e = 0{,}5(e_l + e_r), \quad \delta e = e_i - e_r. \qquad (8)$$

Whence, it follows determination of equation of the motion of aircraft. By this it means that Newton’s equations are of second order hence aerodynamic equation of motion is also equation of second order. For three of the components of the angle velocity (p,q,r), derivatives are of first order by arguments – current time, hence for these three variable states we derive a system of differential equation.

(9) $$\dot{x}_i = F_i\left(x;c\right), \quad 1 \le i \le 3.$$

The two “polar” angels of motion expressed by components of velocity w, v, which are derivatives of first kind related to current flying time. Hence, these two angles are described through equations (9) with indexes $4 \le i \le 5$.

At the end it follows determination of derivatives related to current time of the three Eulers's oriented angles (ϕ, θ, ψ) of motion of aircraft. Its definition is connected with the following linear deformation of the three components of the angle velocity $\left(p, q, r\right)$:

(10) $$\begin{aligned} &\dot{\phi} = p + q.tg\theta.\sin\phi + r.tg\theta.\cos\phi, \\ &\dot{\theta} = 0 + q.\cos\phi.\sin 90^\circ - r.\sin\phi. \end{aligned}$$

The system of equations from (9) with indexes $6 \le i \le 8$ play a role in determinant equation. The mathematical analysis of the motion of aircraft leads to obtaining a system of kind (1) with eight variables and five control parameters.

Equations of the motion of aircraft are three of the components of angle velocity $\left(p, q, r\right)$ for $1 \le i \le 3$ and they are not depending of the three orientated Euler's angles (ϕ, θ, ψ). Beside this, the equations of the two “polar” angles (α, β) for $4 \le i \le 5$ depend on the angels ϕ, θ for existing summands containing gravitation acceleration, divided by modulo of velocity $g \big/ \left|\vec{V}\right|$. For reactive aircraft quantity of velocity $\left|\vec{V}\right|$ is great and this is way quotient $g \big/ \left|\vec{V}\right| \to 0$. From this follows that summands containing gravitate acceleration can be eliminated. As result for reactive aircraft the systems of first five and the last become with divided variables. For this the system of equation of motion for aircraft of reactive and transport kind becomes of the form

(11) $$\dot{x}_i = F_i\left(x_j;c\right), \quad 1 \le i,\, j \le 5,$$

(12) $$\dot{x}_k = F_k\left(x_l;c\right), \quad 6 \le k \le 8, 1 \le l \le 8.$$

3. SOLUTION OF THE SYSTEM OF EQUATION (11)

In case of symmetrical reactive aircraft the system of equation (11) will have one stationary solution $x = 0 \in R^5$ for $c = 0 \in R^5$. For finding other possible solutions the decomposition of the system in Brook Taylor series is used analogously to (2) and (3). From this, the nonlinear differential equation follows

(13) $$\dot{x} = LAM + LWM + NIM \,,$$

where LAM are linear aerodynamic terms (members), LWM - linear control terms and NIM - nonlinear inertial terms from nonlinear differential equation (13).

Linear aerodynamic terms (members) - LAM and linear control terms LWM are determined from

(14) $$LAM = F_i^{\,j} x_j \,, \quad LWM = F_i^{\alpha} c_\alpha \,.$$

Nonlinear inertia terms of nonlinear differential equation (13) are determined from

(15) $$NIM = F_l^{\,jk} x_j x_k \,, \; 1 \le i \ne j \ne k \le 3 \,,$$

where the system of equation is valid

(16) $$\left| \begin{array}{l} F_1^{23} = F_1^{32} = \left(I_{yy} - I_{ZZ} \right) / I_{XX} \,, \\ F_2^{31} = F_2^{13} = \left(I_{ZZ} - I_{XX} \right) / I_{yy} \,, \\ F_3^{12} = F_3^{21} = \left(I_{XX} - I_{yy} \right) / I_{ZZ} \,. \end{array} \right.$$

If all nonlinear terms are neglected except inertial, then the system of nonlinear aerodynamic equations of motion of aircraft of reactive and transport type (the catastrophe function) will take the form

(14) $$\left| \begin{array}{l} \dot{x}_i = \sum\limits_{\alpha} F_i^{\alpha} c_\alpha + \sum\limits_{j=1}^{5} F_i^{\,j} x_j + \sum\limits_{1 \le j < k}^{3} F_i^{\,jk} x_j x_k \,, \; 1 \le i \le 3, \\ \dot{x}_i = \sum\limits_{\alpha} F_i^{\alpha} c_\alpha + \sum\limits_{j=1}^{5} F_i^{\,j} x_j \,, \; 4 \le i \le 5. \end{array} \right.$$

4. CONCLUSIONS

Such derived system of nonlinear differential aerodynamic equations of the motion of aircraft of reactive and transport type (the catastrophe function) expresses the existence of bifurcation set of stationary states of investigated air transport system.

The so synthetic bifurcation set of states is usable for additional investigation for stability and manifold of stationary states.

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Congruences and Reduction Systems in Stable Varieties

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Abstract: *We use abstract reduction systems to describe stable varieties of semigroups and groupoids.*

Keywords: *composition of terms, essential position in terms, stable variety.*

1 Introduction

Let $\mathcal{F}$ be any finite set of *operation symbols*. Let X be a set of variables, and let τ be a *type* which determines the arities of the operation symbols in $\mathcal{F}$. The set $W_\tau(X)$ of *terms of type* τ with variables from X is the smallest set such that $X \subseteq W_\tau(X)$ and if f is an $n-$ary operation symbol, and $t_1,\ldots,t_n \in W_\tau(X)$ then $f(t_1\ldots t_n) \in W_\tau(X)$. $Sub(t)$ denotes the set of all subterms of t. If $f \in \mathcal{F}$, then $f^{\mathcal{A}}$ denotes a $\tau(f)$-ary operation on the set A. An algebra $\mathcal{A} = \langle A; \mathcal{F}^{\mathcal{A}} \rangle$ of type τ is a pair consisting of a set A and an indexed set $\mathcal{F}^{\mathcal{A}}$ of operations, defined on A. If $s,t \in W_\tau(X)$, then the pair $s \approx t$ is called an identity of type τ.

The *inductive composition* $t(r \leftarrow s)$ is a term, obtained by simultaneous replacement of every occurrence of r as a subterm of t by s.

Any term can be regarded as a tree with nodes labeled as the operation symbols and its leaves labeled as variables (see Figure 1). To each occurrence of an operation symbol is assigned a position. The positions are finite sequences (strings) over N_τ, starting with the empty sequence ε for the root position (see Figure 1). $Pos(t)$ denotes the set of positions of t with its natural ordering $p \preceq q \Leftrightarrow p \text{ is a prefix of } q$. Let $sub_t : Pos(t) \to Sub(t)$ be the function which maps each position in a term t to the subterm of t, whose root node occurs at that position. The *positional composition* [5] of t and r on $p \in Pos(t)$ is the term $s := t(p;r)$ obtained from t by replacing the term $sub_t(p)$ by r on the position p, only. Denote by $\preceq^{lex}$ the lexicographical order in N_τ^*.

The variable x_i is called *essential* [4] in t with respect to the algebra $\mathcal{A}$ if there are $n+1$ elements $a_1,\ldots,a_{i-1},a,b,a_{i+1},\ldots,a_n \in A$ such that

$$t^{\mathcal{A}}(a_1,\ldots,a_{i-1},a,a_{i+1},\ldots,a_n) \neq t^{\mathcal{A}}(a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_n).$$

$Ess(t,\mathcal{A})$ denotes the set of all essential variables in t with respect to $\mathcal{A}$.

A variable x_i is said to be $\Sigma-$*essential* [5] in a term t if there is an algebra $\mathcal{A}$, such that $\mathcal{A} \models \Sigma$ and $x_i \in Ess(t,\mathcal{A})$. $Ess(t,\Sigma)$ denotes the set of all $\Sigma-$essential variables in t. The concept of $\Sigma-$essential positions is a natural extension of $\Sigma-$essential variables. If $x_{n+1} \in Ess(t(p;x_{n+1}),\mathcal{A})$, then the position $p \in Pos(t)$ is called *essential* in t with respect to the algebra $\mathcal{A}$ [5]. $PEss(t,\mathcal{A})$ denotes the set of all essential positions in t with respect to $\mathcal{A}$. If $x_{n+1} \in Ess(t(p;x_{n+1}),\Sigma)$ the position $p \in Pos(t)$ is called $\Sigma-$*essential* in t [5]. $PEss(t,\Sigma)$ denotes the set of $\Sigma-$essential positions in t and $PFic(t,\Sigma)$ denotes the set of Σ-*inessential (fictive)* positions of t. The set of Σ-essential subterms of t is defined as follows: $SEss(t,\Sigma) := \{sub_t(p) \mid p \in PEss(t,\Sigma)\}$.

Let $\Sigma S_r^t = \{v \in Sub(t) \mid \Sigma \models r \approx v\}$, $\Sigma P_r^t = \{p \in Pos(t) \mid sub_t(p) \in \Sigma S_r^t\}$ and let $P_r^t = \{p_1,\ldots,p_m\}$ be the set of all the minimal elements in ΣP_r^t with respect to $\preceq$.

Term $\Sigma-$*composition* [5] of t and r by s is defined as follows $t^{\Sigma}(r \leftarrow s) = t$ if $P_r^t = \varnothing$ and $t^{\Sigma}(r \leftarrow s) = t(P_r^t;s)$ if $P_r^t \neq \varnothing$.

Example 1.1. Let $t = f(x_3, f(f(x_1,x_2),x_2))$. Let us consider the variety of rectangular bands $RA = Mod(\Sigma)$, where $\Sigma = \{f(x_1, f(x_2,x_3)) \approx f(f(x_1,x_2),x_3) \approx f(x_1,x_3)\}$. Then the $\Sigma-$essential positions and subterms of t are $PEss(t,\Sigma) = \{\varepsilon,1,2,22\}$ and

$SEss(t,\Sigma)=\{t,x_3,f(f(x_1,x_2),x_2),x_2\}$. The $\Sigma-$essential positions of t are represented by large black circles, in Figure 1.

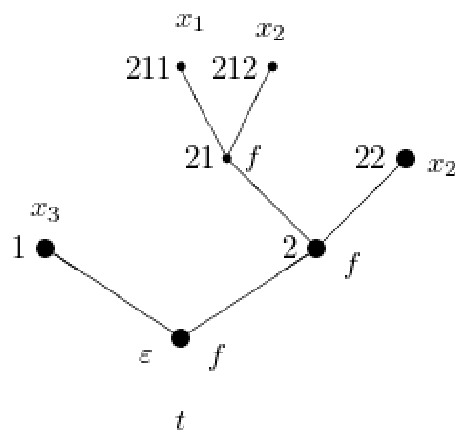

Figure 1: $\Sigma-$essential positions in t from Example 1.1.

2 Stable varieties of type (2)

Definition 2.1 *[1] A set Σ of identities of type τ is $D-$deductively closed if it satisfies the following axioms (some authors call them ``deductive rules", ``derivation rules", ``productions", etc.): D_1 (reflexivity) $t\approx t\in\Sigma$; D_2 (symmetry) $(t\approx s\in\Sigma)\Rightarrow s\approx t\in\Sigma$; D_3 (transitivity) $(t\approx s\in\Sigma)\,\&\,(s\approx r\in\Sigma)\Rightarrow t\approx r\in\Sigma$; D_4 (variable inductive substitution) $(t\approx s\in\Sigma)\,\&\,(r\in W_\tau(X))\Rightarrow t(x\leftarrow r)\approx s(x\leftarrow r)\in\Sigma$; D_5 (term positional replacement)*

$$(t\approx s\in\Sigma)\,\&\,(r\in W_\tau(X))\,\&\,(sub_r(p)=t)\Rightarrow r(p;s)\approx r\in\Sigma.$$

For any set of identities Σ the smallest $D-$deductively closed set containing Σ is called the $D-$closure of Σ and it is denoted by $D(\Sigma)$.

Let Σ be a set of identities of type τ. For $t\approx s\in Id(\tau)$ we say Σ proves $t\approx s$ and write $\Sigma\vdash t\approx s$ if there is a sequence of identities ($D-$deduction) $t_1\approx s_1,\ldots,t_n\approx s_n$, such that each identity belongs to Σ or is a result of applying any of the derivation rules D_1-D_5 to previous identities in the sequence and the last identity $t_n\approx s_n$ is $t\approx s$.

Let $t \approx s$ be an identity and $\mathcal{A}$ be an algebra of type τ. $\mathcal{A} \models t \approx s$ means that the corresponding term operations are equal i.e. $t^{\mathcal{A}} = s^{\mathcal{A}}$.

Let Σ be a set of identities of type τ. Then $\mathcal{A} \models \Sigma$ means that $\mathcal{A} \models t \approx s$ in all $t \approx s \in \Sigma$. For $t, s \in W_\tau(X)$ we say Σ yields $t \approx s$ (write: $\Sigma \models t \approx s$) if, given any algebra $\mathcal{A}$, $\mathcal{A} \models \Sigma \Rightarrow \mathcal{A} \models t \approx s.$ It is well known that $\Sigma \vdash t \approx s \Leftrightarrow \Sigma \models t \approx s$.

In [5] the derivation rules $D_1 - D_5$ are extended up to the globally invariant congruences. A set Σ of identities is ΣR*-deductively closed* if it satisfies the rules D_1, D_2, D_3, D_4 and ΣR_1 *(Σ replacement)* $(t \approx s \in \Sigma)$ & ($r \in SEss(t,\Sigma) \cap SEss(s,\Sigma) \Rightarrow t^{\Sigma}(r \leftarrow u) \approx s^{\Sigma}(r \leftarrow u) \in \Sigma.$

For any set of identities Σ, the smallest $\Sigma R-$deductively closed set containing Σ is called $\Sigma R-$*closure* of Σ and it is denoted by $\Sigma R(\Sigma)$. A set $\Sigma \subseteq Id(\tau)$ is called *the globally invariant congruence* if it is $\Sigma R-$deductively closed [5]. A variety V of type τ is called *stable* if $Id(V)$ is $\Sigma R-$ deductively closed, i.e., $Id(V) = \Sigma R(Id(V))$ is a globally invariant congruence.

The main goal of the present paper is to describe stable varieties of type $\tau = (2)$. First, we prove the following result.

Theorem 2.1 *[6] A variety $V = Mod(\Sigma)$ of semigroups is stable if and only if Σ proves at least one identity among*

$$f(f(x_1, x_2), x_3) \approx f(x_i, x_j), \quad (1)$$

in $i, j \in \{1,2,3\}, i \neq j.$.

Theorem 2.1 shows that no non-trivial variety of groups be stable since (1) can not be identities in any non-trivial group.

Our aim is to use Abstract Reduction Systems (ARS) in description of stable varieties of idempotent groupoids and groupoids.

ARS play an important role in various areas as abstract data type specification , functional programming, automated deductions etc. (see [2]). The concept of ARS also apply to other rewrite systems as string rewrite systems (Thue systems), tree rewrite systems, graph grammars etc.

An ARS is a structure $\mathcal{W} = \langle W_\tau(X), (\rightarrow_i)_{i \in I} \rangle$, where $(\rightarrow_i)_{i \in I}$ is a family of binary relations on $W_\tau(X)$, called *reductions or rewrite relations*. For a reduction $\rightarrow_i$ the transitive and reflexive closure is denoted by $\rightarrow_i^*$.

A term $r \in W_\tau(X)$ is a *normal form* if there is no $v \in W_\tau(X)$ such that $r \rightarrow_i v$.

Many computations, constructions, processes, translations, mappings and so on, can be modelled as stepwise transformations of objects known as rewriting systems. In all different branches of rewriting the basic concepts are the same, and known as termination (guaranteeing the existence of normal forms) and confluence (securing the uniqueness of normal forms).

Let $r = sub_t(p)$, $s = sub_t(q)$ with $\Sigma \models r \approx s$. If $p \prec q$ then we have a redundancy which impact on the valuations of the term t. From D_5 we have $\Sigma \models t \approx t(p;s)$ and all positions and variables which belong to r and not belong to s are redundant. A pair $(p,q) \in Pos(t)^2$ is called *reducible* in $t \in W_\tau(X)$ if: (i) $p \prec q$, $\Sigma \models sub_t(p) \approx sub_t(q)$; (ii) $\Sigma \not\models sub_t(q) \approx sub_t(q')$ for all q' with $q \prec q'$ and $q \prec^{lex} q'$; (iii) there is no $q' \in Pos(t)$ such that $sub_t(q) = sub_t(q')$ and $q \prec^{lex} q'$.

Let $Rd(t,\Sigma)$ denotes the set of all reducible pairs in t. Since $Pos(t) \subset \{1,2\}^*$ we might denote $\overline{\alpha} = 1$ if $\alpha = 2$ and $\overline{\alpha} = 2$ if $\alpha = 1$. for $\alpha \in \{1,2\}$.

We introduce the ARS, defined as follows $\mathcal{W} := \langle W_{(2)}(X), \{\rightarrow_S\}\rangle$, where (i) $t \rightarrow_S r \overset{def}{\Leftrightarrow} r = t(p; sub_t(q))$, where $(p,q) \in Rd(t,\Sigma)$.

The main idea is to reduct the terms downto their normal forms in an identity and then implement the deductive derivations, hopping the normal forms are with low complexity.

Using Newman's Lemma (Theorem 1.2.1. [2]), we prove that the reduction $\rightarrow_S$ has *unique normal form property* (UN), because it is *terminating* (or *strongly normalizing* SN) and *weakly confluent* (or *has weakly Church-Rosser property* WCR).

For each term $t \in W_\tau(X)$ we denote by $Sr(t)$ the unique normal form obtained from t with the reduction $\rightarrow_S$.

Now, we describe a procedure which assigns to each term t of type $\tau = (2)$ its unique normal form $Sr(t)$ under the reduction $\rightarrow_S$.

A reducible pair $(p,q) \in Rd(t,\Sigma)$ is called *minimal* in t if $p \preceq^{lex} p'$ for all $(p',q') \in Rd(t,\Sigma)$. The minimal reducible pair in t is denoted by $minr(t,\Sigma)$.

Procedure 2.1 *Name: Sr;*

Input: a set of identities $\Sigma \subseteq Id(\tau)$, a term $t \in W_\tau(X)$;
Output: the normal form $Sr(t)$ of t under $\to_S$.
begin
$i := 1$;
while $Rd(t,\Sigma) \neq \varnothing$ **do**
begin
$t_i := t(p; sub_t(q)$;
$t := t_i$;
$i := i+1$;
end;
Put $Sr(t) := t$;
end.

If $(p,q) := minr(t,\Sigma)$ then the Procedure 2.1 will output the normal form $Sr(t)$ of the term t under the reduction $\to_S$.

Lemma 2.1 $\Sigma \models t \approx Sr(t)$.

Using the normal form $Sr(t)$ we prove that in the variety of idempotent groupoids $IG = Mod(\Sigma)$ with $\Sigma = \{f(x_1, x_1) \approx x_1\})$ the following is satisfied:

(i) if $s \in Sub(Sr(t))$ then there is $r \in Sub(t)$ such that $\Sigma \models r \approx s$;

(ii) if $t = f(t_1, t_2)$ then $\Sigma \models Sr(t) \approx f(Sr(t_1), Sr(t_2))$ and

(iii) $\Sigma \models Sr(t^\Sigma(r \leftarrow u)) \approx Sr(t)^\Sigma(r \leftarrow u)$ for every $r, t, u \in W_\tau(X)$.

This allows us to prove that the variety $IG = Mod(\{f(x_1, x_1) \approx x_1\})$ of idempotent groupoids is stable and the variety $CG = Mod(\{f(x_1, x_2) \approx f(x_2, x_1)\})$ of all commutative groupoids is stable.

Theorem 2.2 *[6] The varieties* $V = Mod(\{f(f(x_i, x_j), x_k) \approx f(x_m, x_m)\})$ *and* $V = Mod(\{f(x_i, f(x_j, x_k)) \approx f(x_m, x_m)\})$*, for* $i, j, k, m \in \{1,2\}$ *are stable.*

Proposition 2.1 *[6] The varieties* $V = Mod(\{f(f(x_1, x_2), x_3) \approx f(x_i, x_j)\})$ *and* $V = Mod(\{f(x_1, f(x_2, x_3)) \approx f(x_i, x_j)\})$*, for* $i, j \in \{1,2,3\}$*,* $i \neq j$*, are not stable.*

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ONE EXAMPLE OF ANALYTIC FUNCTION ON THE UNIT DISC

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***Abstract**: In this article will be given one example of analytic function on the unit disc which is bounded on almost every radius and doesn't have limit.*

***Keywords:** Analytic function, cluster set, unit disc, family of paths, F_σ -set, set of first category, monotonely bounded path*

1. PRELIMINARIES

In this paper we will use the concepts given in the following definitions.

Definition 1. Let $f(z)$ be a complex valued function on the unit disc $D=\{z \mid |z|<1\}$ and $S \subseteq D$. The cluster set of $f(z)$ on S is the set which contains all the points w of the complex plane, such that there is a sequence $(z_n)_{n\in\mathbb{N}}$ in S so that $\lim\limits_{n\to\infty}|z_n|=1$ and $\lim\limits_{n\to\infty} f(z_n)=w$.

Definition 2. Monotonely bounded path in $D=\{z \mid |z|<1\}$ is a simple continuous curve $u=u(x), 0\le x<1,$ in D, such that $u(0)=0, \lim\limits_{x\to 1}|u(x)|=1, |u(x)|<|u(y)|, for\, x<y$. We are going to use the following notations

$D_r=\{z \mid |z|<r\}, 0<r<1,\ C_r=\{z \mid |z|=r\}, 0<r<1$, and $T=\{z \mid |z|=1\}$.

In the article, the concept of family of paths *F* related to a set $E\subseteq T=\partial D$ is given with the following definition.

Definition 3. Let E be nonempty subset of the unit circle in the complex plane $T=\{z \mid |z|=1\}$, $E\subseteq T=\partial D$. For arbitrary $t\in E$, there is monotonely bounded path s_t in D, such that the intersection of any two of this monotonely bounded paths is the zero and for any $r \in (0,1)$, the 1-1 corespondence $t \leftrightarrow s_t \cap C_r, t \in E$ keeps the cyclic ordering.

We will say that the set $F = \{s_t \mid t \in E\}$ is a family of paths on D, if the following conditions are satisfied:

(i) E is F_σ- set of first category on $T = \{z \mid |z| = 1\}$,

(ii) There is a decomposition of E, $E = \bigcup_{n=1}^{\infty} E_n$, where every E_n is closed and nowhere dense relative to $T = \{z \mid |z| = 1\}$, such that if $T_n = \bigcup_{t \in E_n} s_t$, then $T_n \cap D_r$ is closed and nowhere dense set.

Example 1. Let E be nonempty F_σ set of first category on $T = \{z \mid |z| = 1\}$. For every $t = e^{i\varphi} \in E$ we define $s_t : [0,1) \to D$ with $s_t(r) = r \cdot e^{i\varphi}$. Then the set of radii $F = \{s_t \mid t \in E\}$ is a family of paths relative to E.

2. CONSTRUCTION

Theorem 1. Let $g(z)$ is a continuous complex valued function on D, and $F = \{s_t \mid t \in E\}$ is a family of paths relative to E on D. There exists analytic function $f(z)$ on D, such that for any path from F the cluster sets of $f(z)$, $\operatorname{Re} f(z)$, $\operatorname{Im} f(z)$ coincide with the cluster sets of $g(z)$, $\operatorname{Re} g(z)$, $\operatorname{Im} g(z)$ respectively.

Proof. Let $(r_n)_{n \in \mathbb{N}}$ be a sequence, $0 < r_0 < r_1 < \ldots < r_n < \ldots < 1$ and $\lim_{n \to \infty} r_n = 1$ and $F_0 = D_{r_0} \cup (T_1 \cap C_{r_1})$. Inductively, we define

$$F_n = D_{r_n} \cup \left[\left(\bigcup_{j=1}^{n} T_j \right) \cap \left(D_{r_{n+1}} \setminus D_{r_n} \right) \right] \cup \left(T_{n+1} \cap C_{r_{n+1}} \right), \text{for any } n \in \mathbb{N}$$

From the definition of F, every set F_n is bounded, closed and its complement is connected set. Next, we define inductively two real-valued sequences $\varphi_n(z)$ and $\psi_n(z)$ on F_n and polynomials $P_n(z)$ as follows. Let

$$\varphi_0(z)=\psi_0(z)=0,\ z\in D_{r_0}$$

$$\varphi_0(z)=\operatorname{Re} g(z),\ \psi_0(z)=\operatorname{Im} g(z),\ z\in T_1\cap C_{r_1}.$$

The function $\varphi_0(z)+i\cdot\psi_0(z)$ is continuous on F_0 and analytic in every inner point of F_0. Using the theorem of Mergelyan, there is a polynomial $P_0(z)$, such that $\left|P_n(z)-\left[\varphi_0(z)+i\cdot\psi_0(z)\right]\right|\le 1$, for any $z\in F_0$.

We assume that $n>0$ and we have defined functions $\varphi_{n-1}(z)$ and $\psi_{n-1}(z)$ on F_{n-1} and polynomials $P_0(z),P_1(z),...,P_{n-1}(z)$ such that $\varphi_{n-1}(z)=\operatorname{Re} g(z)$, $\psi_{n-1}(z)=\operatorname{Im} g(z)$, $z\in F_{n-1}\cap C_{r_n}$, and

(1) $$\left|P_0(z)+P_1(z)+...+P_{n-1}(z)-g(z)\right|\le\frac{1}{2^{n-1}},z\in F_{n-1}\cap C_{r_n}$$

Using the theorem of Tietze on (1), it follows that there exist real-valued functions $\xi_n(z)$ and $\eta_n(z)$, continuous on D such that

$$\xi_n(z)=\operatorname{Re}\sum_{j=0}^{n-1}P_j(z),\ \eta_n(z)=\operatorname{Im}\sum_{j=0}^{n-1}P_j(z),\ z\in F_{n-1}\cap C_{r_n}$$

$$\xi_n(z)=\operatorname{Re} g(z),\ \eta_n(z)=\operatorname{Im} g(z),\ z\in F_{n-1}\cap C_{r_{n+1}}$$

(2) $$\left|\xi_n(z)-\operatorname{Re} g(z)\right|\le\frac{1}{2^{n-1}},\ \left|\eta_n(z)-\operatorname{Im} g(z)\right|\le\frac{1}{2^{n-1}},\ z\in\bigcup_{j=0}^{n}T_j,\ r_n<|z|<r_{n+1}$$

Now we put

(3) $$\varphi_n(z)=\operatorname{Re}\sum_{j=0}^{n-1}P_j(z),\psi_n(z)=\operatorname{Im}\sum_{j=0}^{n-1}P_j(z),z\in D_{r_n}$$

(4) $$\varphi_n(z)=\xi_n(z),\psi_n(z)=\eta_n(z),z\in F_n\cap\left(D_{r_{n+1}}\setminus D_{r_n}\right).$$

It is clear that the function $\varphi_n(z)+i\cdot\psi_n(z)$ is continuous on F_n and analytic in the interior of F_n. Using the theorem of Mergelyan once more, there is a polynomial $P_n(z)$, such that

(5) $$\left|P_0(z)+P_1(z)+...+P_n(z)-\left[\varphi_n(z)+i\cdot\psi_n(z)\right]\right|\le\frac{1}{2^n},\ \text{for } z\in F_n.$$

With the last step, we have completed the induction.

Let $f(z)=\sum_{j=0}^{\infty}P_j(z)$ for $z\in D$. If $z\in D_{r_n}$, then from (3) and (5), it follows

(6) $$\left|P_{n+j}(z)\right|\le\frac{1}{2^{n+j}},\ j\in\mathbb{N}.$$

We can conclude that $f(z)$ is analytic on D.

Let t be an arbitrary fixed element of E. We will prove that

(7) $$\lim_{\substack{|z|\to1\\ z\in s_t}}\left(\operatorname{Re}f(z)-\operatorname{Re}g(z)\right)=0$$

From (ii) there exists $m\in\mathbb{N}$, such that $t\in E_m$. Let $k\ge0$, $z\in s_t$ and $r_{m+k}<|z|<r_{m+k+1}$. From the definition of $f(z)$ we get

(8) $$\lim_{\substack{|z|\to1\\ z\in s_t}}\left(\operatorname{Re}f(z)-\operatorname{Re}g(z)\right)\le\gamma_k+\delta_k+\rho_k$$

where $\gamma_k(z)=\left|\operatorname{Re}\sum_{j=0}^{m+k}P_j(z)-\varphi_{m+k}(z)\right|$, $\delta_k(z)=\left|\varphi_{m+k}(z)-\operatorname{Re}g(z)\right|$,

$\rho_k=\sum_{j=m+k+1}^{\infty}P_j(z)$. If we apply (5) for $n=m+k$, we get

(9) $$\gamma_k\le\frac{1}{2^{m+k}}$$

From (4), (2) and (1) for $n = m + k$, we get

(10) $$\delta_k \le \frac{1}{2^{m+k+1}}$$

Finally, if we put $n = m + k + 1$ in (6), we obtain

(11) $$\rho_k \le \sum_{j=0}^{\infty} \frac{1}{2^{m+k+1+j}}$$

Combining (8), (9), (10) and (11), we get

(12) $$\left|\operatorname{Re} f(z) - \operatorname{Re} g(z)\right| \le \sum_{j=m+k-1}^{\infty} \frac{1}{2^j}$$

from which directly follows (7).

Similarly, it can be shown that

(12) $$\lim_{\substack{|z|\to 1 \\ z\in s_t}} \left(\operatorname{Im} f(z) - \operatorname{Im} g(z)\right) = 0.$$

From (7) and (13), we get $\lim\limits_{\substack{|z|\to 1 \\ z\in s_t}} \left(f(z) - g(z)\right) = 0$.

As a consequence we obtain the following theorems.

Theorem 2. There exists analytic function $f(z)$ on D, such that for almost every element of $[0, 2\pi)$, valid are the following

$\lim\limits_{r\to 1} \operatorname{Re} f\left(r \cdot e^{i\theta}\right) = 0$ and $\lim\limits_{r\to 1} \operatorname{Im} f\left(r \cdot e^{i\theta}\right) = +\infty$.

Proof. In the example that we considered earlier, let $E = \bigcup\limits_{n\in\mathbb{N}} P_n$ where every P_n is perfect nowhere dense set with measure $\mu(P_n) = 2\pi - \frac{1}{n}$. Thus, the set E is F_σ- set of first category and measure $\mu(E) = 2\pi$. Now if we apply theorem 1 on the family of paths of example 1 with the continuous function $g(z) = \frac{i}{1-|z|}$ we get the existence of the wanted function.

Theorem 3. There exists a function $F(z)$ which is analytic on D, such that the cluster set of $F(z)$ on almost every radius in D is the unit circle $T=\partial D$. Thus, on almost every radius in D, $F(z)$ is bounded and $\lim\limits_{|z|\to 1} F(z)$ doesn't exist.

Proof. The function $F(z)=e^{f(z)}$ where $f(z)$ is the function of theorem 2 satisfies the conditions of the theorem 3.

Let $s=\left\{r\cdot e^{i\theta} \mid 0\le r<1\right\}$ be a radius on which $\lim\limits_{r\to 1}\operatorname{Re} f\left(r\cdot e^{i\theta}\right)=0$ and $\lim\limits_{r\to 1}\operatorname{Im} f\left(r\cdot e^{i\theta}\right)=+\infty$. Let w be an element of the cluster set of $F(z)$, i.e. there is a sequence $(z_n)_{n\in\mathbb{N}}$ in S so that $\lim\limits_{n\to\infty}|z_n|=1$ and $\lim\limits_{n\to\infty}F(z_n)=w$. It is obvious that

$$|w|=\lim_{n\to\infty}\left|F(z_n)\right|=\lim_{n\to\infty}\left|e^{f(z_n)}\right|=\lim_{n\to\infty}e^{\operatorname{Re} f(z_n)}\left|e^{\operatorname{Im} f(z_n)}\right|=1.$$

Conversely, let $w=e^{i\varphi}\in T=\partial D$. Since $f(z)$ is continuous and $\lim\limits_{r\to 1}\operatorname{Re} f\left(r\cdot e^{i\theta}\right)=0$ for sufficiently big $n_0\in\mathbb{N}$, there exist $z_n\in s$ such that $\operatorname{Im} f(z_n)=2\pi n+\varphi$ for every positive integer n, $n\ge n_0$.

Because $f(z)$ is analytic on D, it has to satisfy $\lim\limits_{n\to\infty}|z_n|=1$ and

$$\lim_{n\to\infty}F(z_n)=\lim_{n\to\infty}e^{f(z_n)}=\lim_{n\to\infty}e^{\operatorname{Re} f(z_n)}\cdot e^{i\cdot\operatorname{Im} f(z_n)}=\lim_{n\to\infty}e^{i\cdot(2\pi n+\varphi)}=e^{i\cdot\varphi}$$

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ABOUT THE CENTER OF GRAVITY OF ZEROES OF POLYNOMIALS

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Abstract: *In this paper, polynomials of one variable are considered, as well as operations and properties connected with center of their gravity.*
Keywords: *Polynomial, center of gravity of a polynomial, straight line of gravity of a polynomial.*
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1. INTRODUCTION

The center of gravity $C(x_c, y_c)$ of n points in the plane with coordinates $M_i(x_i, y_i)$ and masses m_i, i=1, 2,…, n is found [1] by the formulas

(1.1) $$x_c = \frac{\sum_{i=1}^{n}(m_i.x_i)}{\sum_{i=1}^{n} m_i}, \quad y_c = \frac{\sum_{i=1}^{n}(m_i.y_i)}{\sum_{i=1}^{n} m_i}.$$

When points $M_i(x_i, y_i)$, i=1, 2,…, n are with equal masses, for the center of gravity $C(x_c, y_c)$ we get:

(1.2) $$x_c = \frac{x_1 + x_2 + ... + x_n}{n}, \quad y_c = \frac{y_1 + y_2 + ... + y_n}{n}$$

Below we will consider points with equal masses, and we will take into account if there are coinciding points.

If every point $M_i(x_i, y_i)$ is presented as a complex number z_i, i=1, 2,…, $n,$ and the center of gravity of the above system of points $C(x_c, y_c)$ as a complex number z_c, then [1] we have:

(1.3) $$z_c = \frac{z_1 + z_2 + ... + z_n}{n}.$$

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$, $a_n \neq 0$, be an arbitrary polynomial of one variable whose coefficients are complex numbers in the general case.

If the roots (zeroes) of $P(z)$ are z_1, z_2,..., z_n, that is, $P(z_i) = 0$, i=1, 2,…, n, and S^p is the sum of its roots, then according to Viete's formulas, the following relation holds:

(1.4) $$S^p = z_1 + z_2 + ... + z_n = \frac{-a_{n-1}}{a_n}.$$

If there are equal roots among roots of a polynomial, then we take into consideration their multiplicity.

2. MAIN RESULTS

Under the concept of "center of gravity" of a given polynomial of one variable we will understand the center of gravity of its roots.

Center of gravity of the polynomial $F(z)$ will be denoted by z_c^F. The straight line passing through the origin of the coordinate system and $z_c^F \neq 0$ will be called the straight line of the gravity of $F(z)$.

Lemma 2.1. If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$, $a_n \neq 0$ then its center of gravity z_c^P satisfies $z_c^P = \frac{-a_{n-1}}{n a_n}$, and if $z_c^P \neq 0$, then with the polynomial $P(z)$, the straight line of gravity $g: \ z = \lambda z_c^P$, where λ is a real parameter, could be uniquely associated, such that the origin of the coordinate system, center of gravity z_c^P and sum of the roots S^p belong to this straight line.

Proof: If roots of the polynomial $P(z)$ are z_1, z_2,..., z_n, then according to (1.3) we have: $z_c^P = \frac{z_1 + z_2 + ... + z_n}{n}$.

Since according to Viete's formulas (1.4), $S^p = z_1 + z_2 + ... + z_n = \frac{-a_{n-1}}{a_n}$, then $z_c^P = \frac{z_1 + z_2 + ... + z_n}{n} = \frac{-a_{n-1}}{n a_n}$, which was to be demonstrated (Q.E.D.).

From $z_c^P = \frac{z_1 + z_2 + ... + z_n}{n} = \frac{S^P}{n}$ it follows that $S^p = n z_c^P$. When λ =0, λ =1, $\lambda = n$, we obtain that the origin of the coordinate system, center of gravity z_c^P and sum of the roots S^p belong to the straight line of gravity g.

_♦_

From *Lemma 2.1.* it can be seen that the center of gravity of a polynomial $P(z)$ depends only on coefficients a_n and a_{n-1}.

Corollary 2.1. Let $F(z)$ and $H(z)$ be polynomials of degree m and n, respectively, such that m-n≥2. If $Q(z)=a.F(z)+b.H(z)$, $a \neq 0$ and b are complex or real numbers, then polynomials $F(z)$ and $Q(z)$ have the same center of gravity.

Corollary 2.2. If a_{n-1}=0, then the center of gravity of the polynomial $P(z)$ coincides with the origin of the coordinate system.

Corollary 2.3. If the number $\frac{-a_{n-1}}{a_n}$ is real then the center of gravity of the polynomial $P(z)$ lies on the axis OX.

From *Lemma 2.1.* we get:

Corollary 2.4. If $F(z)$ and $H(z)$ are polynomials with the same center of gravity $z_c \neq 0$, where $\deg(F(z))=m$, $\deg(H(z))=n$, then with both $F(z)$ and $H(z)$, the same straight line of gravity h: $z = \lambda z_c$, can be associated, on which the sums of the roots S^F and S^H lie, and

$$|S^F - S^H| = |mz_c - nz_c| = |m - n|.|z_c|.$$

Theorem 2.1. If $F(z)$ and $H(z)$ are polynomials with the same center of gravity, then polynomial $Q(z) = F(z).H(z)$ has the same center of gravity.

Proof: Without loss of generality, let $\deg(F(z))=n$, $\deg(H(z))=m$, and roots of the polynomial $F(z)$ are x_1, x_2,..., x_n, and roots of $H(z)$ are y_1, y_2,..., y_m, respectively. From the assumption of Theorem 2.1 it follows that

$$z_c^F = \frac{x_1 + x_2 + ... + x_n}{n} = z_c^H = \frac{y_1 + y_2 + ... + y_m}{m} = z_c$$

By using the above equality, we have $x_1+x_2+...+x_n=nz_c$ and $y_1+y_2+...+y_m=mz_c$. Roots of the polynomial $Q(z)$ will be the roots of both $F(z)$ and $H(z)$. Therefore, $z_c^Q=\dfrac{x_1+x_2+...+x_n+y_1+....+y_m}{n+m}==\dfrac{nz_c+mz_c}{n+m}=z_c$, which completes the proof.

_♦_

When we determine the center of gravity of the set of zeroes (roots) of several polynomials, for the sake of brevity, we will say that we are looking for the center of gravity of a system of polynomials.

Corollary 2.5. If $F_i(z)$, i=1, 2,…, s, are polynomials with the same center of gravity and $n_i\in N^+$={0, 1, 2,…}, i=1, 2,…, s, are arbitrary natural numbers, then polynomial $Q(z)=[F_1(z)]^{n_1}[F_2(z)]^{n_2}...[F_s(z)]^{n_s}$, $\deg(Q(z))\geq 1$, has the same center of gravity.

Similarly to the proof of Theorem 2.1, prove the correctness of the following proposition:

Proposition 2.1. Let $F(z)$ and $H(z)$ be polynomials of degree m and n, respectively, and centers of gravity z_c^F and z_c^H, respectively, where $z_c^F\neq z_c^H$. If $Q(z)=F(z).H(z)$, then center of gravity z_c^Q divides the segment with ends z_c^F and z_c^H into the ratio $\lambda=\dfrac{n}{m}$, where $z_c^Q=\dfrac{mz_c^F+nz_c^H}{m+n}$.

Theorem 2.2. If $P(z)$ is a polynomial, where $\deg(P(z))>1$, then $P(z)$ and its first derivative $P'(z)$ have the same center of gravity.

Proof: Without loss of generality, let

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0,\ a_n\neq 0.$$

From *Lemma 2.1* we have $z_c^P=\dfrac{-a_{n-1}}{n.a_n}$. Taking into account that $P'(z)=na_nz^{n-1}+(n-1)a_{n-1}z^{n-2}+...+a_1$ and it has (n-1) roots and after applying *Lemma 2.1*, we get $z_c^{P'}=\dfrac{-(n-1)a_{n-1}}{(n-1).na_n}=\dfrac{-a_{n-1}}{na_n}=z_c^P$,

which completes the proof.

_♦_

Since $P''(z)=[P'(z)]'$, then if $P''(z)$ is not a constant, from *Theorem 2.2.* it follows that polynomials $P(z)$, $P'(z)$ and $P''(z)$ have the same center of gravity.

Continuing the above reasoning, we obtain:

Corollary 2.6. If $P(z)$ is a polynomial with $\deg(P(z))=n>1$, then polynomials $P(z)$ and $P^{(k)}(z)$, $k=1,\ldots,n-1$, (k-th derivative of $P(z)$) have the same center of gravity.

From *Theorem 2.2***.** and *Corollary 2.4.* we have**:**

Corollary 2.7. If $P^{(k)}$ and $P^{(q)}$ are k-th and q-th derivative of the polynomial $P(z)$, respectively, where $1\leq k\leq q<\deg(P(z))$, then $|S^{P^{(k)}}-S^{P^{(q)}}|=(q-k)|z_c^P|$, and if $z_c^P\neq 0$, the sums of the roots $S^{P^{(k)}}$ and $S^{P^{(q)}}$ lie on the straight line of the gravity of $P(z)$.

From $\left(\int P(z)dz\right)'=P(z)$ and *Theorem 2.2.* it follows:

Corollary 2.8. If $P(z)$ is a polynomial, $Q_s(z)$ is an s-tuple indefinite integral of $P(z)$, and q is an arbitrary integer, then polynomials $P(z)$ and $Q_s(z)$**,** $s=1, 2,\ldots, q$, have the same center of gravity.

Corollary 2.9. If $P(z)$ is a polynomial with $\deg(P(z))=n$, and $F(z)$ is a polynomial for which there exists a number r, such that

$\deg(F^{(r)}(z)-P(z))\leq n-2$, then polynomials $P(z)$ and $F(z)$ have the same center of gravity.

Corollary 2.10. If $P(z)$ is a polynomial with $\deg(P(z))=n$, and $F(z)$ is a polynomial, obtained after r-tuple integration of the polynomial $H(z)$, where $\deg(F(z)-P(z))\leq n-2$, then polynomials $P(z)$, $H(z)$ and $F(z)$ have the same center of gravity.

Taking into account *Theorem 2.1* and *Theorem 2.2*, we have:

Corollary 2.11. If Ω_{z_c} is a set consisting of all polynomials with center of gravity equal to z_c, then each subset of this set generates a system of roots with the same center z_c, and when $z_c\neq 0$, the sum of the roots lies on the straight line of gravity $h: z=\lambda z_c$.

When $z_c \neq 0$, with the set Ω_{z_c}, the straight line of gravity $h\colon z = \lambda z_c$ could be associated, on which the sum of the roots of elements of Ω_{z_c} are represented by points of the straight line *h*, namely: z_c, $2z_c$, $3z_c$,..., nz_c ,….

The sums of the roots of polynomials, whose center of gravity is equal to μz_c, where μ is a real parameter, will lie on the same straight line of gravity.

Theorem 2.3. If $P(z)$ is a polynomial with $\deg(P(z))=n$, then each rotation and homothety in the plane with center at z_c^P preserves the center of gravity of its roots.

Proof: If the roots of the polynomial $P(z)$ are z_1, z_2,..., z_n, then we have: $z_c^P = \frac{z_1 + z_2 + ... + z_n}{n}$. Let apply both rotation $\rho(z_c^P, \varphi)$ and homothety $h(z_c^P, r)$ to the roots of *P(z)*. When *r*=1, we will have only rotation to the angle φ, and when φ=0, we will have only homothety $h(z_c^P, r)$. Multiplication of the number z by $re^{i\varphi}$ is equivalent to applying homothety $h(O,r)$ and rotation $\rho(O,\varphi)$ to this number *z*.

Application of homothety $h(z_c^P, r)$ and rotation $\rho(z_c^P, \varphi)$ to z is equivalent to successive application to z of translation in the direction (- z_c^P), homothety $h(O,r)$, rotation $\rho(O,\varphi)$ and translation in the direction z_c^P.

If we apply rotation $\rho(z_c^P, \varphi)$ and homothety $h(z_c^P, r)$ to the roots of the polynomial *P(z)*, we get: $\bar{z}_i = (z_i - z_c^P) re^{i\varphi} + z_c^P$, where a point z_i is transformed into a point $\bar{z}_i$, *i*=1, 2,…, n.

About the center of gravity of the points $\bar{z}_i$, *i*=1, 2,…, n we have

$$\frac{\sum_{s=1}^{n} \bar{z}_s}{n} = \frac{\sum_{s=1}^{n} [(z_s - z_c^p) re^{i\varphi} + z_c^p]}{n} = \frac{re^{i\varphi}(z_1 + ... + z_n - nz_c^P) + nz_c^P}{n} =$$

$\frac{re^{i\varphi}(nz_c^P - nz_c^P) + nz_c^P}{n} = z_c^P$, that is, the center of gravity remains the same.

_♦_

When $r = 1$ and $\varphi = \pi$ we have:

Corollary 2.12. If $P(z)$ is a polynomial, then each central symmetry in the plane with center z_c^P preserves its center of gravity.

Corollary 2.13. Each subset of Ω_{z_c} preserves its center of gravity when applying homothety $h(z_c, r)$ and rotation $\rho(z_c, \varphi)$.

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On a mathematical model of cancer invasion

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***Abstract**: The paper is devoted to mathematical model of invasion of tumor cells through a healthy tissue. The model is a system of partial differential equations of reaction-diffusion-chemotaxis type. The system is solved numerically. The role of some of the model parameters is analyzed.*
***Keywords:** Mathematical modeling, Computer simulations, Cancer.*

1. INTRODUCTION

Centuries ago tumors were considered "pathological material", which generally cannot be separated from the body, accumulates, becomes "malignant" and causes death, and if it had spread to other parts of the body, the possibility of finding adequate treatment significantly reduced. 90% of deaths were caused not by the original tumor, but by the side effects of the metastatic cancer [2].

Nowadays there is a significant progress in the field of clinical research related to anticancer medications and treatment of cancer. The detection of cancer at an early stage is crucial for successful treatment. Cancer progression is observed when cells in a part of the body begin to grow uncontrollably. Typically, all kinds of cancers begin with the uncontrolled growth of cancer cells. Unlike normal cells, cancer cells are characterized by rapid reproduction and distribution among tissues. Cancers can be divided into two groups - benign and malignant. We will focus our study on a particular type, namely malignant tumor - solid tumors. We will base our analysis on the mathematical model proposed by Chaplain et al. [1,3]. The model describes the interactions between cancer cells, extracellular matrix (ECM) and matrix degradative enzymes (MDE). Extracellular matrix is a network of proteins and carbohydrates that binds cells together, supports and surrounds cells, regulates cells activities, and give assistance for cell movement. ECM is generally deregulated and becomes disorganized by cancer. Sometimes ECM influences the development of cancer through direct promoting cellular transformation and metastasis.

The mathematical method and the approximation scheme are described in Section 2. In Section 3 results of numerical simulations are presented.

2. MATHEMATICAL MODEL

Our paper is devoted to numerical analysis of the reaction-diffusion-chemotaxis model of tumor invasion and metastases proposed by Chaplain *et al.* [1, 3]:

$$\begin{aligned}
&\frac{\partial n}{\partial t} = d_n \frac{\partial^2 n}{\partial x^2} - \chi \frac{\partial}{\partial x}\left(n \frac{\partial m}{\partial x} \right) + \mu_1 n(1-n-f),\\
(1)\quad &\frac{\partial f}{\partial t} = -\eta m f + \mu_2 (1-n-f),\\
&\frac{\partial m}{\partial t} = d_m \frac{\partial^2 m}{\partial x^2} + \alpha n - \beta m.
\end{aligned}$$

The unknown function $n = n(x;t)$ denotes the cancer cells density, $f = f(x;t)$ denotes the ECM density and $m = m(x;t)$ - the MDE concentration. They depend on the space variable x which belongs to the scaled domain $\Omega = [0,1]$ of tissue, and time t. The meaning of the parameters of the model is as follows. The diffusion coefficients d_n and d_m describe the random motility of cancer cells and MDE's respectively, the chemotactic coefficient χ - the directional migration of cancer cells toward higher concentrations of soluble MDE [3], μ_1 denotes the rate of proliferation of cancer cells, η - the destruction rate of ECM due to activity of MDE, μ_2 - the rate of renewal of ECM, α - the production of MDE's by cancer cells and β - the degradation rate of MDE's. The model parameters are supposed to be non-negative.

The system (1) is supplemented by the zero-flux boundary conditions

$$(2)\quad \frac{\partial n}{\partial x}(x,t) = \frac{\gamma}{d_n} n(x,t) \frac{\partial m}{\partial x}(x,t), \frac{\partial m}{\partial x}(x,t) = 0, x = 0,1,$$

and initial conditions

$$(3)\quad n(x,0) = n_0(x), f(x,0) = f_0(x), m(x,0) = m_0(x).$$

We suppose that the functions $n_0(x)$, $f_0(x)$ and $m_0(x)$ are non-negative and are not identically zero.

The model described by Eqs.(1)-(3) is analyzed numerically in [4] where a non-standard finite difference method for its approximation is proposed. The authors define the uniform mesh $\overline{\omega}_h = \{x_i = ih, i = 0,\dots,N\}$ over $[0,1]$ for a positive number N and $h=1/N$ as well as grid points over $[0,T]$, denoted by $t^{k+1}=t^k+\tau_k$, $k=0,1,2,\ldots,$ $t^0=0$. They approximate the third equation of the model by the implicit with respect to m scheme:

(4) $$\frac{m_i^{k+1}-m_i^k}{\tau}=d_m m_{\bar{x}x,i}^{k+1}+\alpha n_i^k-\beta m_i^{k+1}, i=1,\ldots,N-1,$$

which gives:

(5) $$\left(\frac{1}{\tau}+2\frac{d_m}{h^2}+\beta\right)m_i^{k+1}-\frac{d_m}{h^2}(m_{i-1}^{k+1}+m_{i+1}^{k+1})=\frac{1}{\tau}m_i^k+\alpha n_i^k,\ i=1,\ldots,N-1$$

Hereafter, m_i^k denotes the approximation of function $m(x,t)$ at point (x_i,t^k). Boundary conditions (2) are approximated by the equations:

(6) $$\left(\frac{1}{\tau}+2\frac{d_m}{h^2}+\beta\right)m_0^{k+1}-\frac{2d_m}{h^2}m_1^{k+1}=\frac{1}{\tau}m_0^k+\alpha n_0^k,\ i=0$$

(7) $$-\frac{2d_m}{h^2}m_{N-1}^{k+1}+\left(\frac{1}{\tau}+2\frac{d_m}{h^2}+\beta\right)m_N^{k+1}=\frac{1}{\tau}m_N^k+\alpha n_N^k,\ i=N$$

It is shown that the algebraic system (5)-(7) possesses a unique non-negative solution with respect to the unknown variables $m_i^{k+1}, i=0,\ldots,N$, independently of the mesh parameters [4].

The second equation of the model is approximated in [4] by the implicit with respect to f scheme

$$\frac{f_i^{k+1}-f_i^k}{\tau}=-\eta m_i^k f_i^{k+1}+\mu_2[f_i^k(1-n_i^k-f_i^k)^+-f_i^{k+1}(1-n_i^k-f_i^k)^-], i=0,\ldots,N$$

where for an arbitrary value f_i^k of the mesh function f

$$f^+=\max\{0,f\},\ f^-=\max\{0,-f\}\Rightarrow f=f^+-f^-\Rightarrow$$

(8) $$f_i^{k+1}=\frac{f_i^k[1+\tau\mu_2(1-n_i^k-f_i^k)^+]}{1+\tau[\eta m_i^{k+1}+\mu_2(1-n_i^k-f_i^k)^-]},\ i=0,\ldots,N$$

In [4] a nonlocal approximation of $\frac{\partial n}{\partial x}$ taking values at different time levels in the standard finite-difference approximations formulae is proposed. This gives unconditionally positive explicit scheme for $n_i^{k+1}, i=0,\ldots,N$:

$$\frac{n_i^{k+1}-n_i^k}{\tau}=d_n\frac{n_{i+1}^{k+1}-2n_i^{k+1}+n_{i-1}^{k+1}}{h^2}-\gamma\frac{\partial}{\partial x}\left(n\frac{\partial m}{\partial x}\right)_i^{k+1}+M_i^+-M_i^-,\ i=1,\dots,N-1$$

where

$$\frac{\partial}{\partial x}\left(n\frac{\partial m}{\partial x}\right)_i^k=\frac{n_i^{k+1}-n_{i-1}^{k+1}}{h}(m_{\dot{x},i}^k)^+-\frac{n_{i+1}^k-n_i^{k+1}}{h}(m_{\dot{x},i}^k)^-+n_i^{k+1}(m_{\bar{x}x,i}^k)^+-n_i^k(m_{\bar{x}x,i}^k)^-\ ,$$

$$M_i^+=\mu_1 n_i^k(1-n_i^k-f_i^{k+1})^+,\ M_i^-=\mu_1 n_i^{k+1}(1-n_i^k-f_i^{k+1})^-,$$

$$m_{\bar{x}x,i}^k=\frac{m_{i+1}^k-2m_i^k+m_{i-1}^k}{h^2}$$

and similar equations hold for the boundary points.

3. NUMERICAL RESULTS

The aim of this Section is to present the results of our numerical simulations with respect to the role of parameter μ_1 describing the proliferation rate of cancer cells. The algebraic system approximating the model has been solved with Matlab. Following [4], we choose the initial conditions as:

$$n(x,0)=\exp\left(\frac{-x^2}{\varepsilon}\right),f(x,0)=1-n(x,0),m(x,0)=0.5n(x,0),\forall x\in[0,1].$$

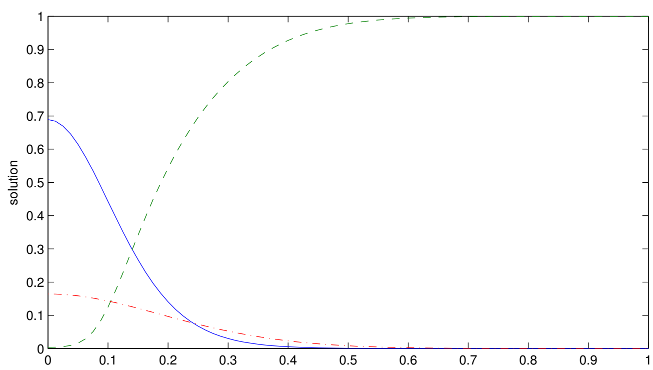

Figure 1. Values of cancer cell density (solid), ECM density (dashed), and MDE concentration (dashdot). Solutions with parameter value $\mu_1=0.1$ at time $t=2$.

The obtained approximate solution for cancer cells, ECM and MDE for values of time $t=2$ and $t=10$ are presented in Figures (1)-(2) for value $\mu_1=0.1$ and in Figures (3)-(4) for $\mu_1=0.5$. The values of the remaining parameters have been set to: $\alpha=0.05$, $\beta=0.15$, $\chi=0.05$, $\eta=10$, $d_n=0.005$, $d_m=0.01$, $\mu_2=2.5$, $\varepsilon=0.01$, $N=80$.

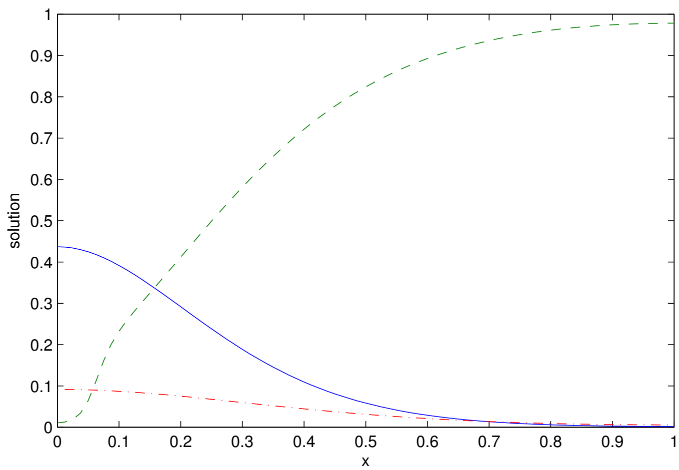

Figure 2. Values of cancer cell density (solid), ECM density (dashed), and MDE concentration (dashdot). Solutions with parameter value $\mu_1=0.1$ at time $t=10$.

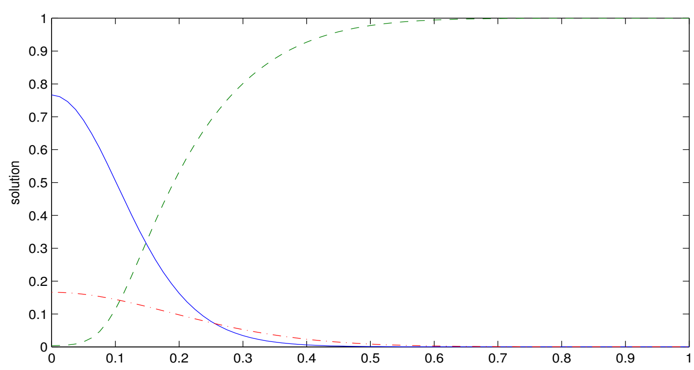

Figure 3. Values of cancer cell density (solid), ECM density (dashed), and MDE concentration (dashdot). Solutions with parameter value $\mu_1=0.5$ at time $t=2$.

The obtained results of simulations show that for low values of cancer proliferation ($\mu_1=0.1$) the ECM is able to fight successfully and decrease the density of tumor cells (Figures (1)-(2)). For higher rates of cancer proliferation ($\mu_1 = 0.5$) the cluster of tumor cells decreases and is able to invade the healthy tissue possibly leading to metastases, which are very dangerous and can be the cause of patient death.

The results of our analysis confirm the important role of the ability of cancer cells to proliferate and illustrate the efficacy of the proposed numerical scheme for approximation of a complicated system of PDEs, which preserves the positivity property of the solutions.

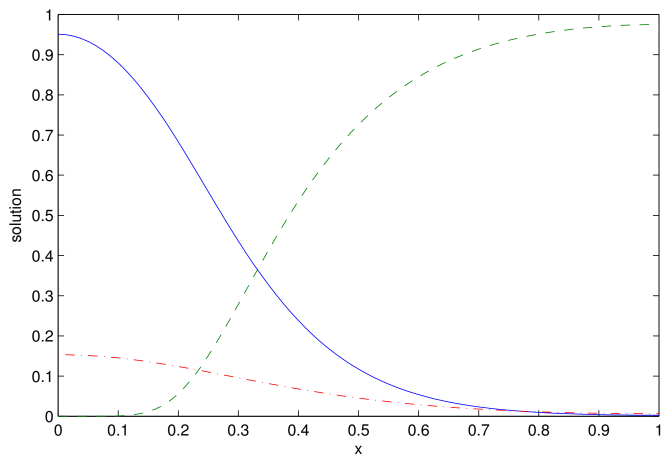

Figure 4. Values of cancer cell density (solid), ECM density (dashed), and MDE concentration (dashdot). Solutions with parameter value $\mu_1 = 0.5$ at time $t = 10$.

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Investigation of the regions of stability of Gear's Implicit m-Step Methods

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***Abstract**: Equalities guaranteeing approximation of order p for m-step Gear's methods are derived. Stability regions for the methods of order 1 to 6 are constructed. It is shown that the Gear's method of order 7 is unstable.*
***Keywords:** Differences methods, stability region, stiff problems.*

1. INTRODUCTION

Mathematical modeling of processes in different areas of science leads to solving systems of complicated, and often nonlinear ordinary differential equations. Such systems can not be solved by known analytical methods and therefore they have to be solved with appropriately selected numerical methods.

In this paper we focus on Gear's method for solving initial value problems, used in particular by the solver ode15s in Matlab [3]. The general form of equation that will be considered is:

(1) $\frac{du(x)}{dx} = f(x,u)$, where $x > x_0, \quad u(x_0) = u_0$.

We assume that for solving the problem (1) the implicit m-step Gear's method is used:

(2) $\sum_{k=0}^{m} \frac{a_k y_{n-k}}{h} = f_n$,

where $n = m,(m+1),...$. The initial values $y_0, y_1,..., y_{m-1}$ are given [1].

The error resulting from the substitution of $u(x_n)$ by y_n is $z_n = y_n - u(x_n)$.

After substituting in equation (2) y_j by $y_j = u_j + z_j$ for $j = \overline{n,(n-m)}$ and adding and subtracting from the right to equality $f(x_n, u_n)$ it is obtained:

(3) $\sum_{k=0}^{m} \frac{a_k z_{n-k}}{h} = \psi_{n-m} + \varphi_{n-m}, \quad n = m,(m+1),...,$

where

(4) $\psi_{n-m} = -\sum_{k=0}^{m} \frac{a_k u_{n-k}}{h} + f(x_n, u_n)$

(5) $\varphi_{n-k} = f(x_n, u_n - z_n) - f(x_n, u_n)$.

The function ψ_{n-m} is called the error of approximation.

2. CONDITIONS ENSURING THE ORDER P OF THE ERROR OF THE APPROXIMATION OF THE METHOD

If we assume that the function $f(x,u)$ satisfies the Lipschitz condition for the second argument i.e.
$|f(x,u_1) - f(x,u_2)| \le L|u_1 - u_2|$, for all values x, u_1, u_2 of the area concerned, we get the following assessment: $|\varphi_{n-m}| \le L|z_n|$.

We expand $u_{n-k} = u(x_n - kh)$ in Taylor series of the order p about $x = x_n$, and obtain

(6) $u_{n-k} = \sum_{l=0}^{p} \frac{(-kh)^l u^{(l)}(x_n)}{l!} + O(h^{p+1}),\ k = \overline{1.m}$.

After substituting (6) in (4) we get:

(7) $$\psi_{n-k} = -\sum_{k=0}^{m} \frac{a_k}{h} \left(\sum_{l=0}^{p} \frac{(-kh)^l u^{(l)}(x_n)}{l!} \right) + u'(x_n) + O(h^p) =$$
$$= \sum_{l=0}^{p} \left(\sum_{k=0}^{m} \frac{a_k (-kh)^l u^{(l)}(x_n)}{h.l!} \right) + u'(x_n) + O(h^p).$$

After elementary transformations we obtain:

(8) $$\psi_{n-m} = -\left(\sum_{k=0}^{m} \frac{a_k}{h} \right) u(x_n) + \sum_{k=1}^{m} k a_k u'(x_n) + u'(x_n) +$$
$$\sum_{l=2}^{p} \left(\sum_{k=1}^{m} \left(-k^l h^{l-1} a_k \frac{u^{(l)}(x_n)}{l!} \right) \right) + O(h^p)$$
.

Therefore, the approximation error is of the order $O(h^p)$ if

(9) $\sum_{k=0}^{m} a_k = 0$; $\sum_{k=1}^{m} k a_k = -1$ and $\sum_{k=1}^{m} k^l a_k = 0$, $l = \overline{2, p}$.

The equations (9) form a system of (p +1) equations in (m +1) unknowns. The system is not redefined if the inequality $p \le m$ is satisfied. This means that the order of approximation of m-step difference method does not exceed m.

3. PARTICULAR DEFINITIONS OF STABILITY OF DIFFERENCE METHODS FOR SOLVING STIFF SYSTEM OF DIFFERENTIAL EQUATIONS

Definition: We call region of stability of a given multistep method the set of points in the complex plane, defined by the complex variable $\sigma = h\lambda$ for which this method applied to the modeled problem $y' = \lambda y, \quad y(x_0) = y_0$
is stable (i. e. prevents the growth of error).

Beyond the usually used definition of stability (i. e. the modulus of all roots of its characteristic equation to be smaller than or equal to one) for stiff systems of differential equations can be used also other more specific definitions.

Definition: A linear multistep method is said to be A-stable if its region of absolute stability, contains the whole of the left-hand complex half-plane [3].

For asymptotically stable systems of equations, all Re λ <0, which means that A - stable method is absolutely stable if the solution of the differential equation is stable.

To solve the stiff systems ODE it is appropriate to use the A-stable methods because their stability does not impose restrictions on the step h.

Definition: A linear multistep method is said to be A(α)-stable, $\alpha \in \left(0, \frac{\pi}{2}\right)$, if its region of absolute stability contains the infinite wedge

$$|\arg(-\mu)| < \alpha, \quad \mu = h\lambda .$$

(α is the angle between the tangent to the curve which is the limit of the area of stability, built at the point (0,0), and the negative direction of the real axis) [2].

According to that definition when $\alpha = \frac{\pi}{2}$, $A(\alpha)$- stability coincides with A-stability.

It is proved that: The order of an A-stable implicit linear multistep method cannot exceed 2. No explicit linear multistep method is $A(\alpha)$-stable.

By using equations (9) we obtain the following numerical schemes for the Gear's method:

1. For m =1 $y_n = y_{n-1} + h.f(x_n, y_n)$.

It coincides with the implicit Euler method

2. For m=2

$$y_n = \frac{4}{3} y_{n-1} - \frac{1}{3} y_{n-2} + \frac{2}{3} .h.f(x_n, y_n).$$

3.For m =3

$$y_n = \frac{18}{11} y_{n-1} - \frac{9}{11} y_{n-2} + \frac{2}{11} y_{n-3} + \frac{6}{11} h.f(x_n, y_n).$$

4. For m =4

$$\frac{25y_n - 48y_{n-1} + 36y_{n-2} - 16y_{n-3} + 3y_{n-4}}{12} = hf(x, y)$$

5. For m=5

$$\frac{137y_n - 300y_{n-1} + 300y_{n-2} - 200y_{n-3} + 75y_{n-4} - 12y_{n-5}}{60} = hf(x_n, y_n)$$

6. For m=6

$$\frac{147y_n - 360y_{n-1} + 450y_{n-2} - 400y_{n-3} + 225y_{n-4} - 72y_{n-5} + 10y_{n-6}}{60} = hf(x_n, y_n)$$

7. For m=7

$$\frac{1089}{420} y_n - \frac{294}{42} y_{n-1} + \frac{441}{42} y_{n-2} - \frac{490}{42} y_{n-3} + \frac{1225}{420} y_{n-4} + \frac{588}{420} y_{n-5} + \frac{7}{6} y_{n-6} -$$
$$- \frac{8}{21} y_{n-7} = hf(x_n, y_n)$$

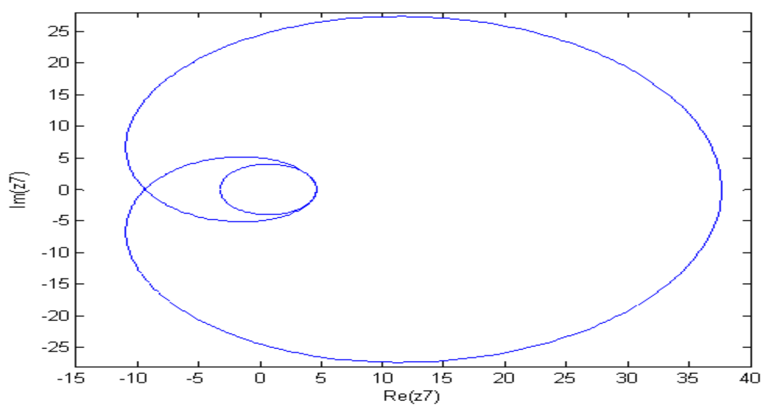

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Fig. 1. Regions of stability of Gear's implicit m-step methods for m=7

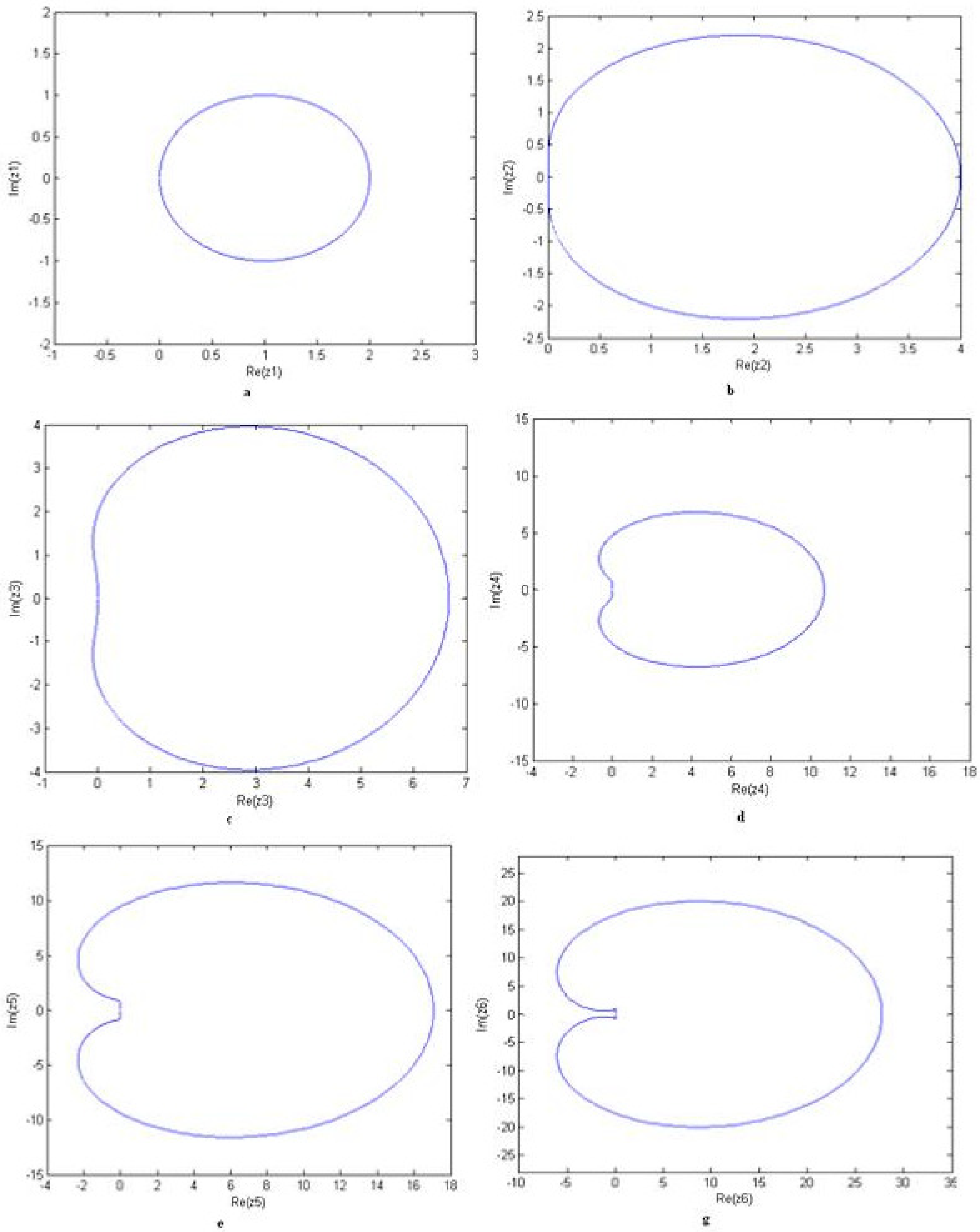

Fig. 2. Regions of stability of Gear's implicit m-step methods for:
a) m=1, b) m=2, c) m=3, d) m=4, e) m=5, f) m=6

The regions of stability are outside the closed curves.

4. CONCLUSIONS

For solving systems of stiff differential equations one must apply Gear's methods of order lower than or equal to six. For $m \geq 7$ the backward differentiation methods of order m of the kind considered here are not stable and are therefore irrelevant from practical point of view. Gear's numerical method is validated in general in solving all kinds of complicated nonlinear differential equations.

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On the weighted $(W(b);\gamma)-$diaphony of the generalized Van der Corput sequence

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We study a new weighted version of the $b-$adic diaphony, the so-called weighted $(W(b);\gamma)-$ diaphony. The exact order $O(1N)$ of the $(W(b);\gamma)-$diaphony of the generalized Van der Corput sequence is obtained. The obtained order is smaller than the order of the $b-$adic diaphony of this sequence.

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Key words: *Uniform distribution of sequences, Diaphony, Generalized Van der Corput sequence*

1 Introduction

Let $s \geq 1$ be a fixed integer which will denote the dimension everywhere in the paper. Following Kuipers and Niederreiter [6] we will recall the concept of uniform distribution of sequences in the $s-$dimensional unit cube $[0,1)^s$. So, let $\xi = (x_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For an arbitrary subinterval J of $[0,1)^s$ with a volume $\mu(J)$ and each integer $N \geq 1$ we set $A_N(\xi;J) = \#\{n : 0 \leq n \leq N-1, x_n \in J\}$. The sequence ξ is called uniformly distributed in $[0,1)^s$ if the equality $\lim_{N\to\infty} N^{-1} A_N(\xi;J) = \mu(J)$ holds for every subinterval J of $[0,1)^s$.

So, let $b \geq 2$ be an arbitrary integer. Following Chrestenson [1], for an arbitrary non-negative integer k and a real $x \in [0,1)$, with the $b-$adic representations $k=\sum_{i=0}^{\nu}k_i b^i$ and $x=\sum_{i=0}^{\infty}x_i b^{-i-1}$, where for $i\geq 0$ $x_i, k_i \in \{0,1,\dots,b-1\}$, $k_\nu \neq 0$ and for infinitely many values of i $x_i \neq b-1$, the $k-$th function of Walsh ${}_b wal_k : [0,1)\to\mathbb{C}$ is defined as

$${}_b wal_k(x) = e^{2\pi i b(k_0x_0+k_1x_1+\dots+k_\nu x_\nu)}.$$

Let $\mathbb{N}_0$ denote the set of the non-negative integer numbers. For an arbitrary vector $k=(k_1,\dots,k_s)\in\mathbb{N}_0^s$ the $k-$th function of Walsh is defined as

$${}_b wal_k(x)=\prod_{h=1}^{s} {}_b wal_{k_h}(x_h),\ \ x=(x_1,\dots,x_s)\in[0,1)^s.$$

The set $W(b)=\{{}_b wal_{\vec{k}} : \vec{k}\in\mathbb{N}_0{}^s\}$ is called Walsh functional system in base b.

In [2] the authors introduced a new weighted version of the $b-$adic diaphony. We will remind the definition of this diaphony.

Definition 1 *Let $\gamma=(\gamma_1,\gamma_2,\dots,\gamma_s)$, where $\gamma_1\geq\gamma_2\geq\dots\geq\gamma_s>0$ be an arbitrary vector of weights. For each integer $N\geq 1$ the weighted $(W(b);\gamma)-$ diaphony $F_N(W(b);\gamma;\xi)$ of the first N elements of the sequence $\xi=(x_n)_{n\geq 0}$ of points in $[0,1)^s$ is defined as*

$$F_N(W(b);\gamma;\xi)=\left(C^{-1}(b;s;\gamma)\sum_{k\in\mathbb{N}_0^s,\,k\neq 0}R(b;\gamma;k)\left|1N\sum_{n=0}^{N-1} {}_b wal_k(x_n)\right|^2\right)^{12},$$

where for each $k=(k_1,\dots,k_s)\in\mathbb{N}_0^s$ the coefficient $R(b;\gamma;k)=\prod_{h=1}^{s}\rho(b;\gamma_h;k_h)$, for a real $\gamma>0$ and an arbitrary integer $k\geq 0$ the coefficient $\rho(b;\gamma;k)$ is given as

$$\rho(b;\gamma;k)$$

$$=\begin{cases}1, & \text{if}\ \ k=0,\\ \gamma b^{-4a}, & \text{if}\ \ k=k_{a-1}b^{a-1},\ k_{a-1}\in\{1,\dots,b-1\},\ a\geq 1,\\ \gamma b^{-2(a+j)}+\gamma b^{-4a}, & \text{if}\ \ k=k_{a-1}b^{a-1}+k_{j-1}b^{j-1}+k_{j-2}b^{j-2}+\dots+k_0,\\ & k_{a-1},\ k_{j-1}\in\{1,\dots,b-1\},\ 1\leq j\leq a-1,\ a\geq 2,\end{cases}$$

and the constant $C(b;s;\gamma)=\prod_{h=1}^{s}\left[1+\gamma_h(b+2)b(b+1)(b^2+b+1)\right]-1.$

It is shown the fact that the weighted $(W(b);\gamma)-$diaphony is a numerical measure for uniform distribution of sequences, in sense, that the sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the equality $\lim_{N\to\infty}F_N(W(b);\gamma;\xi)=0$ holds for an arbitrary vector γ with nonincreasing positive weights.

2 On the weighted $(W(b);\gamma)-$diaphony of the gene-ralized Van der Corput sequence

In the next definition we will present the concept of an important class of well-distributed sequence, the so-called generalized Van der Corput sequence.

Definition 2 *Let $\Sigma=(\sigma_i)_{i\geq 0}$ be a sequence of permutations of the set $\{0,1,\ldots,\ b-1\}$. If an arbitrary non-negative integer n has the $b-$adic representation $n=\sum_{i=0}^{\infty}a_i(n)b^i$, then we replace $S_b^{\Sigma}(n)=\sum_{i=0}^{\infty}\sigma_i(a_i(n))b^{-i-1}$. The sequence $S_b^{\Sigma}=\{S_b^{\Sigma}(n)\}_{n\geq 0}$ is called a generalized Van der Corput sequence.*

The sequence S_b^{Σ} has been defined by Faure [3], and it is an example of a class of sequences which have very well distribution and an effective constructive algorithm. If $\Sigma=I-$ the identity $\sigma_i(a)=a$ the obtained sequence S_b^I is the well known sequence of Halton [5]. When $b=2,$ the sequence S_2^I is the original Van der Corput [7] sequence.

In [4] it has been proved that the $b-$adic diaphony of the generalized Van der Corput sequence has an exact order $O\left(N^{-1}\sqrt{\log N}\right)$.

In the next theorem we will expose upper and lower bounds of the $(W(b);\gamma)-$ diaphony of the generalized Van der Corput sequence.

Theorem 1 *Let $S_b^{\Sigma}=(S_b^{\Sigma}(n))_{n\geq 0}$ be an arbitrary generalized Van der Corput sequence. Then we have the following:*

(i) There is a positive constant $C(b)$, depending only on the base b, such that for each integer $N \geq 1$ the weighted $(W(b);\gamma)-$diaphony $F_N(W(b);\gamma;S_b^{\Sigma})$ of the sequence S_b^{Σ} satisfies the inequality

$$(NF_N(W(b);\gamma;S_b^{\Sigma}))^2 \leq C(b)\left(1-1b^2(N+1)^2\right);$$

(ii) (an asymptotic behavior) For each integer $N \geq 1$ the weighted $(W(b);\gamma)-$ diaphony $F_N(W(b);\gamma;S_b^{\Sigma})$ of the sequence S_b^{Σ} satisfies

$$F_N(W(b);\gamma;S_b^{\Sigma}) \in O(1N);$$

(iii) There is a positive constant $C^{'}(b)$, depending only on the base b, such that for arbitrary integer $\nu > 0$ the inequality

$$F_N(W(b);\gamma;S_b^{\Sigma}) \geq C^{'}(b) \cdot 1N$$

holds for infinitely many values of N of the form $N = b^{\nu} + 1$.

The low bound of the $(W(b);\gamma)-$diaphony gives us the exactness of the order of this diaphony, so, the $(W(b);\gamma)-$ diaphony of an arbitrary generalized Van der Corput sequence has an exact order $O(1N)$.

When we compare the orders of the both kinds of the diaphony- the $b-$adic diaphony and the weighted $(W(b);\gamma)-$diaphony, we conclude that the $(W(b);\gamma)$ -diaphony of an arbitrary generalized Van der Corput sequence has an order which is smaller than the order of the $b-$adic diaphony of this sequence.

Proof of Theorem 1: (i) Let $N \geq 1$ be an arbitrary integer and $\gamma > 0$ be a real number. According to Definition 2 in the one dimensional case we have:

$$(b+2)b(b+1)(b^2+b+1)\left(NF_N(W(b);\gamma;S_b^{\Sigma}\right)^2 (1)$$

$$= \sum_{a=1}^{\infty} b^{-4a} \sum_{k_{a-1}=1}^{b-1} \left| \sum_{n=0\,b}^{N-1} wal_{k_{a-1}b^{a-1}}(S_b^{\Sigma}(n)) \right|^2$$

$$+ \sum_{a=2}^{\infty} b^{-2a} \sum_{k_{a-1}=1}^{b-1} \sum_{j=1}^{a-1} b^{-2j} \sum_{k_{j-1}=1}^{b-1} \sum_{k=k_{a-1}b^{a-1}+k_{j-1}b^{j-1}}^{k_{a-1}b^{a-1}+(k_{j-1}+1)b^{j-1}-1} \left| \sum_{n=0\,b}^{N-1} wal_k(S_b^{\Sigma}(n)) \right|^2$$

$$+\sum_{a=2}^{\infty} b^{-4a} \sum_{k_{a-1}=1}^{b-1} \sum_{j=1}^{a-1} \sum_{k_{j-1}=1}^{b-1} \sum_{k=k_{a-1}b^{a-1}+k_{j-1}b^{j-1}}^{k_{a-1}b^{a-1}+(k_{j-1}+1)b^{j-1}-1} \left| \sum_{n=0_b}^{N-1} wal_k(S_b^{\Sigma}(n)) \right|^2$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3.$$

In [4] the following estimation of the Walsh trigonometric sum with respect to the generalized Van der Corput sequence it has been proved: For arbitrary integers $\alpha \geq 0$ and $\nu \geq 0$ we define the function

$$\delta_{b^{\alpha}}(\nu) = \begin{cases} 1, & \text{if } \alpha \geq \nu, \\ 0, & \text{if } \alpha < \nu. \end{cases}$$

Let an arbitrary integer $N \geq 1$ has the $b-$adic representation $N = a_1 b^{\nu_1} + a_2 b^{\nu_2} + \ldots + a_t b^{\nu_t}$, where $\nu_1 > \nu_2 > \ldots > \nu_t \geq 0$ and for $1 \leq \tau \leq t$ $a_\tau \in \{1, 2, \ldots, b-1\}$. Let $k \geq 1$ be an arbitrary integer with the $b-$adic representation $k = k_1 b^{\alpha_1} + k_2 b^{\alpha_2} + \ldots + k_g b^{\alpha_g}$, where $\alpha_1 > \alpha_2 > \ldots > \alpha_g \geq 0$ and for $1 \leq \tau \leq g$ $k_\tau \in \{1, 2, \ldots, b-1\}$. Then the following inequality holds

$$\left| \sum_{n=0_b}^{N-1} wal_k(S_b^{\Sigma}(n)) \right| \leq \sum_{p=1}^{t} a_p b^{\nu_p} \delta_{b^{\alpha_g}}(\nu_p). (2)$$

We will use the next scheme of addition: If $(a_{pq})_{p,q=1}^{n}$ is a symmetric matrix, then the equality $\sum_{p=1}^{n} \sum_{q=1}^{n} a_{pq} = 2\sum_{p=1}^{n} \sum_{q=1}^{p} a_{pq} - \sum_{p=1}^{n} a_{pp}$ holds.

From (2) and the above scheme of addition we have the following results:

$$\Sigma_1 \leq C_1(b) \sum_{p=1}^{t} b^{-2\nu_p}, (3)$$

where $C_1(b) = 2b^4 - 2b^3 - b^2 - b - 1(b+1)(b^2+1)(b^2+b+1)$.

We will estimate the sum Σ_2. For an arbitrary integer $j \geq 2$ and $\alpha = 0, 1, \ldots, j-2$ we define the set

$$B(j;\alpha)$$

$$= \{ l : l = l_{j-2} b^{j-2} + \ldots + l_\alpha b^\alpha, \alpha \leq \tau \leq j-2, l_\tau \in \{0, 1, \ldots, b-1\}, l_\alpha \neq 0 \}.$$

It is obviously that $|B(j;\alpha)| = (b-1) b^{j-2-\alpha}$. Then from (2) we have:

$$\Sigma_2 = \sum_{a=2}^{\infty} b^{-2a} \sum_{k_{a-1}=1}^{b-1} \sum_{j=1}^{a-1} b^{-2j} \sum_{k_{j-1}=1}^{b-1} \left[\left| \sum_{n=0_b}^{N-1} wal_{k_{a-1}b^{a-1}+k_{j-1}b^{j-1}}(S_b^{\Sigma}(n)) \right|^2 \right. \quad (4)$$

$$\left. + \sum_{\alpha=0}^{j-2} \sum_{l \in B(j;\alpha)} \left| \sum_{n=0_b}^{N-1} wal_{k_{a-1}b^{a-1}+k_{j-1}b^{j-1}+l}(S_b^{\Sigma}(n)) \right|^2 \right]$$

$$\leq (b-1)^2 \sum_{a=2}^{\infty} b^{-2a} \sum_{j=1}^{a-1} b^{-2j} \left[\sum_{p=1}^{t} \sum_{q=1}^{t} a_p a_q b^{\nu_p+\nu_q} \delta_{b^{j-1}}(\nu_p) \delta_{b^{j-1}}(\nu_q) \right.$$

$$\left. + \sum_{\alpha=0}^{j-2} \sum_{l \in B(j;\alpha)} \sum_{p=1}^{t} \sum_{q=1}^{t} a_p a_q b^{\nu_p+\nu_q} \delta_{b^{\alpha}}(\nu_p) \delta_{b^{\alpha}}(\nu_q) \right] = (b-1)^2 \sum_{a=2}^{\infty} b^{-2a} \sum_{j=1}^{a-1} b^{-2j} \times$$

$$\left[2 \sum_{p=1}^{t} \sum_{q=1}^{p} a_p a_q b^{\nu_p+\nu_q} \delta_{b^{j-1}}(\nu_p) \delta_{b^{j-1}}(\nu_q) - \sum_{p=1}^{t} a_p^2 b^{2\nu_p} \delta_{b^{j-1}}(\nu_p) \right]$$

$$+ (b-1)^3 b^2 \sum_{a=2}^{\infty} b^{-2a} \sum_{j=1}^{a-1} b^{-2j} \sum_{\alpha=0}^{j-2} b^{j-\alpha} \times$$

$$\left[2 \sum_{p=1}^{t} \sum_{q=1}^{p} a_p a_q b^{\nu_p+\nu_q} \delta_{b^{\alpha}}(\nu_p) \delta_{b^{\alpha}}(\nu_q) - \sum_{p=1}^{t} a_p^2 b^{2\nu_p} \delta_{b^{\alpha}}(\nu_p) \right]$$

$$\leq 2(b-1)^4 \sum_{p=1}^{t} b^{\nu_p} \sum_{q=1}^{p} b^{\nu_q} \sum_{a=2}^{\infty} b^{-2a} \sum_{j=1}^{a-1} b^{-2j} \delta_{b^{j-1}}(\nu_p) \delta_{b^{j-1}}(\nu_q)$$

$$-(b-1)^2 \sum_{p=1}^{t} b^{2\nu_p} \sum_{a=2}^{\infty} b^{-2a} \sum_{j=1}^{a-1} b^{-2j} \delta_{b^{j-1}}(\nu_p)$$

$$+ 2(b-1)^5 b^2 \sum_{p=1}^{t} b^{\nu_p} \sum_{q=1}^{p} b^{\nu_q} \sum_{a=2}^{\infty} b^{-2a} \sum_{j=1}^{a-1} b^{-j} \sum_{\alpha=0}^{j-2} b^{-\alpha} \delta_{b^{\alpha}}(\nu_p) \delta_{b^{\alpha}}(\nu_q)$$

$$-(b-1)^3 b^2 \sum_{p=1}^{t} b^{2\nu_p} \sum_{a=2}^{\infty} b^{-2a} \sum_{j=1}^{a-1} b^{-j} \sum_{\alpha=0}^{j-2} b^{-\alpha} \delta_{b^{\alpha}}(\nu_p)$$

$$= 2(b-1)^4 \sum_{p=1}^{t} b^{\nu_p} \sum_{q=1}^{p} b^{\nu_q} \sum_{a=\nu_q+2}^{\infty} b^{-2a} \sum_{j=\nu_q+1}^{a-1} b^{-2j}$$

$$-(b-1)^2 \sum_{p=1}^{t} b^{2\nu_p} \sum_{a=\nu_p+2}^{\infty} b^{-2a} \sum_{j=\nu_p+1}^{a-1} b^{-2j}$$

$$+2(b-1)^5 b^2 \sum_{p=1}^{t} b^{\nu_p} \sum_{q=1}^{p} b^{\nu_q} \sum_{a=\nu_q+3}^{\infty} b^{-2a} \sum_{j=\nu_q+2}^{a-1} b^{-j} \sum_{\alpha=\nu_q}^{j-2} b^{-\alpha}$$

$$-(b-1)^3 b^2 \sum_{p=1}^{t} b^{2\nu_p} \sum_{a=\nu_p+3}^{\infty} b^{-2a} \sum_{j=\nu_p+2}^{a-1} b^{-j} \sum_{\alpha=\nu_p}^{j-2} b^{-\alpha} \le C_2(b) \sum_{p=1}^{t} b^{-2\nu_p},$$

where $C_2(b) = 2b^7(b-1)(b^3+2b^2+b+1(b+1)^2(b^2+b+1)$. Here we assume that, when $j=1$, then $\sum_{\alpha=0}^{j-2}\sum_{l\in B(j;\alpha)} \cdot = 0$.

By using a technique, similar to one, used for estimation of Σ_2, we have:

$$\Sigma_3 \le C_3(b) \sum_{p=1}^{t} b^{-2\nu_p}, (5)$$

where
$C_3(b) = 2b^7 - b^5 - 2b^4 - 3b^3 - 3b^2 - 2b - 3b(b+1)^2(b^2+1)^2(b^2+b+1)^2$.

From (1), (3), (4) and (5) we obtain that

$$\left(NF_N(W(b);\gamma;S_b^{\Sigma}\right)^2 \le C''(b) \sum_{p=1}^{t} b^{-2\nu_p}, (6)$$

where $C''(b) = b(b+1)(b^2+b+1)b + 2(C_1(b)+C_2(b)+C_3(b))$.

By using the $b-$adic representation of $N = \sum_{p=1}^{t} a_p b^{\nu_p}$ we have that $N \ge \sum_{p=0}^{t-1} b^p = b^t - 1b - 1$, so $b^t \le b(N+1)$ and

$$\sum_{p=1}^{t} b^{-2\nu_p} \le \sum_{p=0}^{t-1} b^{-2p} = b^2 b^2 - 1\left(1 - 1b^{2t}\right) \le b^2 b^2 - 1\left(1 - 1b^2(N+1)^2\right). (7)$$

From (6) and (7) we obtain (i) of Theorem 1 with $C(b) = C''(b) \cdot b^2 b^2 - 1$.

The part (ii) of Theorem 1 is a direct consequence from the upper bound of the $(W(b);\gamma)-$ diaphony of the sequence S_b^{Σ}.

(iii) For an arbitrary integer $\nu > 0$ we put $N = b^{\nu} + 1$. According to Definition 1 the $(W(b);\gamma)-$diaphony satisfies the low bound

$$\gamma(b+2)b(b+1)(b^2+b+1)(NF_N(W(b);\gamma;S_b^{\Sigma})^2 (8)$$

$$\geq \sum_{k=1}^{b-1} \rho(b;\gamma;k) \left| \sum_{n=0_b}^{b^{\nu}} wal_k(S_b^{\Sigma}(n)) \right|^2.$$

We will use the equality $\sum_{n=0_b}^{b^{\nu}} wal_k(S_b^{\Sigma}(n)) = \sum_{n=0_b}^{b^{\nu}-1} wal_k(S_b^{\Sigma}(n)) +_b wal_k(S_b^{\Sigma}(b^{\nu}))$. For each integer k, such that $1 \leq k \leq b-1$ we have that $\alpha_g = 0$, and from (2) the equality $\sum_{n=0_b}^{b^{\nu}-1} wal_k(S_b^{\Sigma}(n)) = 0$ holds. Then we have that $\left| \sum_{n=0_b}^{b^{\nu}} wal_k(S_b^{\Sigma}(n)) \right| = 1$. From (8) we obtain the inequality

$$\gamma(b+2)b(b+1)(b^2+b+1)(NF_N(W(b);\gamma;S_b^{\Sigma}))^2 \geq \sum_{k=1}^{b-1} \gamma b^{-4} = \gamma(b-1)b^4.$$

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On the weighted $(W(b);\gamma)-$diaphony of the generalized Zaremba-Halton net

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We study a new weighted version of the $b-$adic diaphony, the so-called weighted $(W(b);\gamma)-$ diaphony. For an arbitrary integer $\nu \geq 3$ the exact order $O\left(\log b^{\nu} b^{2\nu}\right)$ of the $(W(b);\gamma)-$diaphony of the generalized Zaremba-Halton net is obtained.

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Key words: *Diaphony, Generalized Zaremba-Halton net*

1 Introduction

Let $s \geq 1$ be a fixed integer which will denote the dimension everywhere in the paper. Let $\xi = (x_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For an arbitrary subinterval J of $[0,1)^s$ with a volume $\mu(J)$ and each integer $N \geq 1$ we set $A_N(\xi;J) = \#\{n : 0 \leq n \leq N-1, x_n \in J\}$. The sequence ξ is called uniformly distributed in $[0,1)^s$ if the equality $\lim_{N\to\infty} N^{-1} A_N(\xi;J) = \mu(J)$ holds for every subinterval J of $[0,1)^s$.

The quantitative theory of the uniform distribution of sequences $[0,1)^s$ studies the numerical measures for the irregularity of their distribution. Some recently introduced $b-$adic versions of the diaphony are analytical measures for the irregularity of the distribution of sequences.

The Walsh functions in base b will be the main tool of our study. So, let $b \geq 2$ be an arbitrary integer. Following Chrestenson [1], for an arbitrary non-negative integer k and a real $x \in [0,1)$, with the $b-$adic representations $k = \sum_{i=0}^{\nu} k_i b^i$ and $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where for $i \geq 0$ $x_i, k_i \in \{0,1,\ldots,b-1\}$, $k_\nu \neq 0$ and for infinitely many values of i $x_i \neq b-1$, the $k-$th function of Walsh ${}_b wal_k : [0,1) \to \mathbb{C}$ is defined as

$$ {}_b wal_k(x) = e^{2\pi i b(k_0 x_0 + k_1 x_1 + \ldots + k_\nu x_\nu)}. $$

The set $W(b) = \{{}_b wal_k : k = 0,1,\ldots\}$ is called Walsh functional system in base b.

Let $\mathbb{N}_0$ denote the set of the non-negative integer numbers. For an arbitrary vector $k = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ the $k-$th function of Walsh is defined as

$$ {}_b wal_k(x) = \prod_{h=1}^{s} {}_b wal_{k_h}(x_h), \ \ x = (x_1,\ldots,x_s) \in [0,1)^s. $$

In [2] the authors introduced a new weighted version of the $b-$adic diaphony. We will remind the definition of this diaphony:

Definition 1 *Let* $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, *where* $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_s > 0$ *be an arbitrary vector of weights. For each integer* $N \geq 1$ *the weighted* $(W(b);\gamma)-$ *diaphony* $F_N(W(b);\gamma;\xi)$ *of the first* N *elements of the sequence* $\xi = (x_n)_{n\geq 0}$ *of points in* $[0,1)^s$ *is defined as*

$$ F_N(W(b);\gamma;\xi) = \left(C^{-1}(b;s;\gamma) \sum_{k\in\mathbb{N}_0^s,\, k\neq 0} R(b;\gamma;k) \left| 1N \sum_{n=0}^{N-1} {}_b wal_k(x_n) \right|^2 \right)^{12}, $$

where for each vector $k = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ $R(b;\gamma;k) = \prod_{h=1}^{s} \rho(b;\gamma_h;k_h)$, for a real $\gamma > 0$ and an arbitrary integer $k \geq 0$ the coefficient $\rho(b;\gamma;k)$ is given as

$$ \rho(b;\gamma;k) $$

$$ = \begin{cases} 1, & \text{if } \ k = 0, \\ \gamma b^{-4a}, & \text{if } \ k = k_{a-1}b^{a-1}, \ k_{a-1} \in \{1,\ldots,b-1\}, \ a \geq 1, \\ \gamma b^{-2(a+j)} + \gamma . b^{-4a}, & \text{if } \ k = k_{a-1}b^{a-1} + k_{j-1}b^{j-1} + k_{j-2}b^{j-2} + \ldots + k_0, \\ & k_{a-1}, \ k_{j-1} \in \{1,\ldots,b-1\}, 1 \leq j \leq a-1, \, a \geq 2, \end{cases} $$

and the constant $C(b;s;\gamma) = \prod_{h=1}^{s} \left[1 + \gamma_h (b+2)b(b+1)(b^2+b+1) \right] - 1.$

2 On the weighted $(W(b);\gamma)-$diaphony of the generalized Zaremba-Halton net

Let the reals $x, y \in [0,1)$ have the $b-$adic representations $x=\sum_{i=0}^{\infty} x_i b^{-i-1}$ and $y=\sum_{i=0}^{\infty} y_i b^{-i-1}$. Then, we put $x \oplus_b y = \sum_{i=0}^{\infty} [x_i + y_i \ (mod \ \ b)] b^{-i-1}$.

Here, we will recall a general constructive approach of Stoilova [6] to construct a class of two-dimensional nets. For this purpose, let $\alpha, \beta \in \{0,1,\ldots,b-1\}$ be fixed $b-$adic digits and $\nu > 0$ be an arbitrary integer. For each j, such that $0 \le j \le \nu - 1$, let $\mu_j \equiv \alpha \ j + \beta \ (mod \ \ b)$. So, we generate the digits μ_j by using a linear recurrence procedure. Let us set $\mu = 0.\mu_0 \mu_1 \ldots \mu_{\nu-1}$.

Following Halton [3], for an arbitrary non-negative integer n with the $b-$adic representation $n=\sum_{i=0}^{\infty} a_i(n) b^i$ let $\xi_b(n)=\sum_{i=0}^{\infty} a_i(n) b^{-i-1}$ mean the general term of the sequence of Halton. For $0 \le n \le b^\nu - 1$ we put $\eta_{b,\nu}(n) = n b^\nu$ and $\xi'_{b,\nu}(n) = \xi_b(n) \oplus_b \mu$.

Definition 2 *For arbitrary $b-$adic digits α and β and an arbitrary integer $\nu > 0$, the generalized Zaremba-Halton net in base b, composed of b^ν points is defined as*

$$Z_{b,\nu}^{\alpha,\beta} = \{(\eta_{b,\nu}(n), \xi'_{b,\nu}(n)) : 0 \le n \le b^\nu - 1\}.$$

When $\alpha = 1$ and $\beta = 0$, the net $Z_{b,\nu}^{1,0}$ have been introduced by Warnock [7]. When $b = 2$, $\alpha = 1$ and $\beta = 1$, the obtained net $Z_{2,\nu}^{1,1}$ is the original net of Zaremba-Halton [4]. If $\alpha = 0$ and $\beta = 0$, the obtained net $Z_{2,\nu}^{0,0}$ is the original net R_ν of Roth [5].

In the next theorem we will give upper and low bounds of the weighted $(W(b);\gamma)-$ diaphony of the generalized Zaremba-Halton net.

Theorem 1 *Let $\gamma = (\gamma_1, \gamma_2)$ be an arbitrary vector of weights. For arbitrary digits $\alpha, \beta \in \{0,1,\ldots,b-1\}$ and an arbitrary integer $\nu > 0$ let $Z_{b,\nu}^{\alpha,\beta}$*

be the corresponding generalized Zaremba-Halton net. Then, for each integer $\nu \geq 3$ *the weighted* $(W(b);\gamma)-$ *diaphony* $F\left(W(b);\gamma;Z_{b,\nu}^{\alpha,\beta}\right)$ *of the net* $Z_{b,\nu}^{\alpha,\beta}$ *satisfies the inequalities*

$$\gamma_1\gamma_2\Big[(b-1)^2 2b^4\nu^2 b^{-2\nu}-(b-1)^2(5b^4-4b^2-2)2b^8\nu b^{-2\nu}$$

$$+(b-1)^2(3b^4-4b^2-2)b^8 b^{-2\nu}\Big]\leq\Big[C^{'}(b;\gamma)b^\nu F\left(W(b);\gamma;Z_{b,\nu}^{\alpha,\beta}\right)\Big]^2$$

$$\leq\gamma_1\gamma_2\Big[(b-1)^2 2b^4\nu^2 b^{-2\nu}+(4b^2-8b+5)\nu b^{-2\nu}+3(b-1)^3\nu b^{-4\nu}+\nu b^{-6\nu}\Big]$$

$$+\Big[(\gamma_1+\gamma_2)2b^3+2b^2+3(b+1)^2+\gamma_1\gamma_2(5b^3+4b^2-11b+6)\Big]b^{-2\nu}$$

$$+\gamma_1\gamma_2\Big[(b-1)^3(b+17)b^{-4\nu}+(2b^3-5b^2+4b)b^{-6\nu}\Big],$$

where the constant

$$C^{'}(b;\gamma)=(\gamma_1+\gamma_2)(b+2)b(b+1)(b^2+b+1)+\gamma_1\gamma_2(b+2)^2b^2(b+1)^2(b^2+b+1)^2.$$

Theorem 1 shows that the weighted $(W(b);\gamma)-$diaphony of the generalized Zaremba-Halton net has an exact order $O\left(\log b^\nu b^{2\nu}\right)$. In [16] it has been proved that the $b-$adic diaphony of the generalized Zaremba-Halton net has an exact order $O\left(\sqrt{\log b^\nu}\, b^\nu\right)$. Hence, the weighted $(W(b);\gamma)-$diaphony of the generalized Zaremba-Halton net has an order which is smaller than the order of the $b-$adic diaphony of this net.

Corollary 1 will give the exact constant in the exact order of the weighted $(W(b);\gamma)-$ diaphony of the generalized Zaremba-Halton net.

Corollary 1 *Under the conditions of Theorem 2 the following equality*

$$\lim_{\nu\to\infty} b^{2\nu}F\left(W(b);\gamma;Z_{b,\nu}^{\alpha,\beta}\right)\log b^\nu=b-1b^2\log b\sqrt{\gamma_1\gamma_2 2C^{'}(b;\gamma)}$$

holds.

3 Proof of Theorem 1

To prove Theorem 1 we need of the following two lemmas, where some useful upper bounds are obtained.

Lemma 1 *Let* $\gamma_1,\gamma_2>0$ *be arbitrary reals. Then for each integer* $\nu>0$ *the following inequalities*

$$\sum_{k=1}^{\infty}\rho(b;\gamma_1;k)\left|\sum_{n=0}^{b^{\nu}-1}{}_b\, wal_k(\eta_{b,\nu}(n))\right|^2 \le \gamma_1 2b^3+2b^2+3(b+1)^2 b^{-2\nu}$$

and

$$\sum_{l=1}^{\infty}\rho(b;\gamma_2;l)\left|\sum_{n=0}^{b^{\nu}-1}{}_b\, wal_l(\xi'_{b,\nu}(n))\right|^2 \le \gamma_2 2b^3+2b^2+3(b+1)^2 b^{-2\nu}$$

hold.

Lemma 2 *For an arbitrary vector $\gamma=(\gamma_1,\gamma_2)$ of weights and for each integer $\nu\ge 3$ the following inequality*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\rho(b;\gamma_1;k)\rho(b;\gamma_2;l)\left|\sum_{n=0}^{b^{\nu}-1}{}_b\, wal_k(\eta_{b,\nu}(n))_b\, wal_l(\xi'_{b,\nu}(n))\right|^2$$

$$\le \gamma_1\gamma_2\left[(b-1)^2 2b^4\nu^2 b^{-2\nu}+(4b^2-8b+5)\nu b^{-2\nu}+3(b-1)^3\nu b^{-4\nu}+\nu b^{-6\nu}\right.$$

$$\left.+(5b^3+4b^2-11b+6)b^{-2\nu}+(b-1)^3(b+17)b^{-4\nu}+(2b^3-5b^2+4b)b^{-6\nu}\right]$$

holds.

Proof of Theorem 1: Let $\gamma=(\gamma_1,\gamma_2)$ be an arbitrary vector of weights. According to Definition 1 for the $(W(b);\gamma)-$diaphony of the generalized Zaremba-Halton net we have the equality

$$\left[C'(b;\gamma)b^{\nu}F\left(W(b);\gamma;Z_{b,\nu}^{\alpha,\beta}\right)\right]^2$$

$$=\sum_{k=1}^{\infty}\rho(b;\gamma_1;k)\left|\sum_{n=0}^{b^{\nu}-1}{}_b\, wal_k(\eta_{b,\nu}(n))\right|^2+\sum_{l=1}^{\infty}\rho(b;\gamma_2;l)\left|\sum_{n=0}^{b^{\nu}-1}{}_b\, wal_l(\xi'_{b,\nu}(n))\right|^2$$

$$+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\rho(b;\gamma_1;k)\rho(b;\gamma_2;l)\left|\sum_{n=0}^{b^{\nu}-1}{}_b\, wal_l(\eta_{b,\nu}(n))_b\, wal_l(\xi'_{b,\nu}(n))\right|^2.$$

According to the results of Lemmas 1 and 2 from the above equality we obtain that

$$\left[C'(b;\gamma)b^{\nu}F\left(W(b);\gamma;Z_{b,\nu}^{\alpha,\beta}\right)\right]^2$$

$$\leq \gamma_1\gamma_2\left[(b-1)^2 2b^4\nu^2 b^{-2\nu} + (4b^2-8b+5)\nu b^{-2\nu} + 3(b-1)^3\nu b^{-4\nu} + \nu b^{-6\nu}\right]$$

$$+\left[(\gamma_1+\gamma_2)2b^3 + 2b + 3(b+1)^2 + \gamma_1\gamma_2(5b^3+4b^2-11b+6)\right]b^{-2\nu}$$

$$+\gamma_1\gamma_2\left[(b-1)^3(b+17)b^{-4\nu} + (2b^3-5b^2+4b)b^{-6\nu}\right].$$

The obtained results about the upper bound of the $(W(b);\gamma)-$ diaphony of the generalized Zaremba-Halton net, where the main order $\nu^2 b^{-2\nu}$ is obtained, give us the idea, how to obtain a lower bound of the $(W(b);\gamma)-$diaphony of this net. So, we will use the inequality

$$\left[C'(b;\gamma)b^{\nu}F\left(W(b);\gamma;Z_{b,\nu}^{\alpha,\beta}\right)\right]^2$$

$$\geq \gamma_1\gamma_2(b-1)^2 b^{2\nu}\sum_{a=2}^{\nu-1}\sum_{j=1}^{a-1}\left(b^{-2(a+j)}+b^{-4a}\right)\left(b^{-2(2\nu+2-j-a)}+b^{-4(\nu+1-j)}\right)$$

$$\geq \gamma_1\gamma_2\left[(b-1)^2 2b^4\nu^2 b^{-2\nu} - (b-1)^2(5b^4-4b^2-2)2b^8\nu b^{-2\nu}\right.$$

$$\left.+(b-1)^2(3b^4-4b^2-2)b^8 b^{-2\nu}\right].$$

Theorem 1 is finally proved.

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Tools Selection for Design and Development of an Expert System for Social Area Domain

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Abstract: *An expert system is a type of application program that makes decisions or solves problems in a particular field by using knowledge and analytical rules defined by experts in that field. Books and manuals have a tremendous amount of knowledge but a human has to read and interpret the knowledge for it to be used. Taking this into consideration it can be said that a human cannot perform a complex problem because of the different disadvantages (being unreliable, speed, and not enough memory capacity) he/she might be facing. For this case Humans can design expert systems providing knowledge of solving a specified problem. This article describes the technology of building of an expert system of social area domain and its characteristics, explores the circumstances under which the expert system could be useful in social area.*

Keywords: *Expert systems, Quality of life*

1. INTRODUCTION

Standard of living and quality of life are often referred to in discussions about the economic and social well-being of countries and their residents, but what is the difference between the two? The definitions of these terms overlap in some areas, depending on whom you ask. It is more than just a matter of semantics; in fact, knowing the difference can affect how you evaluate a country where you might be looking to invest some money.

The main difference between standard of living and quality of life is that the former is more objective, while the latter is more subjective. Standard of living factors such as Gross Domestic Product, poverty rate and environmental quality, can all be measured and defined with numbers [4] [6] [7], while quality of life factors like equal protection of the law, freedom from discrimination and freedom of religion, are more difficult to measure and are particularly qualitative. Both indicators are flawed, but they can help us get a general picture of what life is like in a particular location at a particular time. Ferrell, who has carried out a large research programme on pain and quality of life, defined quality of life as well-being covering four areas: quality of life is physical, mental, social and spiritual well-being [2]. The World Health Organization defines Quality of life as „an individual's perception of their position in life in the context of the culture and value systems in which they live and in relation to their goals, expectations, standards and concerns. It is a broad ranging concept affected in a complex way by the person's physical health, psychological state, personal beliefs, social relationships and their relationship to salient features of their environment" [5]. "The Economist", for example, produces an index that attempts to rate the quality of life in various countries.

Determinants of quality of life according to Econonist.com [8].

The nine quality-of-life factors, and the indicators used to represent these factors, are: Material wellbeing: GDP per person, at ppp in $; Health: Life expectancy at birth, years; Political stability and security: Political stability and security ratings; Family life: Divorce rate (per 1,000 population), converted into index of 1 (lowest divorce rates) to 5 (highest); Community life: Dummy variable taking value 1 if country has either high rate of church attendance or trade-union membership; zero otherwise; Climate and geography: Latitude, to distinguish between warmer and colder climes; Job security: Unemployment rate, %; Political freedom: Average of indices of political and civil liberties. Scale of 1 (completely free) to 7 (unfree); Gender equality: Ratio of average male and female earnings, latest available data.

2. PROBLEM DEFINITION

Quality of life can be considered as daily living enhanced by wholesome food and clean air and water, enjoyment of unfettered open spaces and bodies of water, conservation of wildlife and natural resources, security from crime, and protection from radiation and toxic substances. It may also be used as a measure of the energy and power a person is endowed with that enable him or her to enjoy life and prevail over life's challenges irrespective of the handicaps he or she may have.

The majority of this article focuses on the necessity of the knowledge based expert system which can be designed to carry the intelligence and information found in the expert's knowledge and provide such kinds of knowledge for other members of the social area for problem solving purposes – evaluation of quality of life. For this area most of the problems which require expert system might seem easy to be solved by a professional, but we develop an expert system, which is used for social area problems for which there is no single “correct” solution which can be encoded in a predictable algorithm.

Our purpose is to design and develop a sophisticated expert system which is capable of making an evaluation of quality of life by taking into account real-world uncertainties.

3. BASIC DEFINITIONS, METHODOLOGY AND TOOLS FOR EXPERT SYSTEMS

Expert Systems are computer programs that are derived from a branch of computer science research called Artificial Intelligence (AI). AI's scientific goal is to understand intelligence by building computer programs that exhibit intelligent behaviour. It is concerned with the concepts and methods of symbolic inference, or reasoning, by a computer, and how the knowledge used to make those inferences will be represented inside the machine.

Knowledge is almost always incomplete and uncertain. To deal with uncertain knowledge, a rule may have associated with it a confidence factor or a weight. The set of methods for using uncertain knowledge in combination with uncertain data in the reasoning process is called reasoning with uncertainty. An important subclass of methods for reasoning with uncertainty is called "fuzzy logic," and the systems that use them are known as "fuzzy systems."

Tools, Shells, and Skeletons: The inference engine is the program that locates the appropriate knowledge in the knowledge base, and infers new knowledge by applying logical processing and problem-solving strategies. Building expert systems by using shells offers significant advantages. A system can be built to perform a unique task by entering into a shell all the necessary knowledge about a task domain. The inference engine that applies the knowledge to the task at hand is built into the shell.

The development of expert system for an evaluation of quality of life is a new information technology derived from Artificial Intelligent research using ESTA (Expert System Text Animation) System. The proposed expert system contains knowledge about different indicators for measuring quality of life. The system is to be developed in the ESTA (Expert System shell for Text Animation) which is Visual Prolog 7.3 Application [9]. The knowledge for this very system will be acquired by domain experts, texts and other related sources.

4. TECHNOLOGY FOR DEVELOPMENT OF EXPERT SYSTEMS

The process of development of an expert system is based on traditional technologies. It can be divided into six stages, which are relatively independent from the domain. The sequence of stages reflects the general idea of creating the optimum product. It is not fixed. It allows to an each stage from the development process of expert system to be submitted new ideas, which may affect the previous decision, and even lead to their processing.

We are in the process of development of an expert system that is to be developed in ESTA Application. It is designed for assisting in interpretation of the knowledge about quality of life and provided adequate evaluation of quality of life.

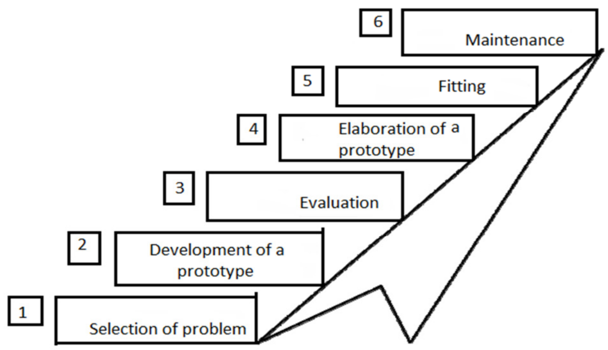

Fig. 1: The stages of development of expert systems

Selection of a problem: To develop an expert system, first we need to identify the problem and understand the major characteristics of the problem that we have to solve in the expert system. The input problem for our system is the indicators of quality of life. The input problem is structured for the system. The expert module recognizes it as a pattern and forwards for processing for providing evaluation of quality of life.

Knowledge Acquisition: Acquisition process in this expert system will have the following modules - Interactive Expert Module, Expert System Program and Coordinating Module for knowledge database.

In the interactive expert module, the domain specific expertise knowledge is acquired from human experts. The acquired knowledge is analyzed and then processed to obtain a best conclusion for the problem. Next the knowledge is transferred to the information system experts to be verified for converting into expert system program. The process is continued until the best conclusion of the problem is obtained. Once the knowledge, acquired either from domain experts or domain resources, is verified by the information system experts, it is transferred from interactive expert module to the expert system program module for converting into expert system program. For our system, expert knowledge has been acquired from standard literature, related to the social area and indicators for quality of life.

Conceptualization of knowledge**:** Conceptualization allows to be revealed the structure of knowledge in the domain, namely the following are determined: terminology; list with basic concepts and their attributes; relationships between concepts; structure of input and output information; strategy for decision making; constraints of the strategy.

Development of a prototype: The prototype of system is a shortened version of the expert system, which is designed to verify the correctness of the encoding facts, the relationships and the strategies of expert reasoning.

Structure model: When the domain is complex it is necessary the structure (architecture) of expert system to be designed in advance. The usage of CASE tools for creating UML-diagrams is possible. This allows faster and more efficient design, early detection and correction of errors that could affect the final result negatively.

Implementation: The implementation is the process of development of a prototype of an expert system with all building blocks. In the design and development of this expert system, we are using the shell ESTA. ESTA has the explanation facilities of the questions in the knowledge base and for the given advice. ESTA contains the rules represented in its own syntax for its knowledge base. It consists of the inbuilt facilities to write the rules that build the knowledge base.

Knowledge representation is the last phase of the knowledge base development. In the representation of knowledge into knowledge base, the knowledge acquired from knowledge acquisition process is represented into structured form. There are many approaches for representing knowledge into the knowledge base. Such representation in ESTA is the rule based representation in logical paradigm of simple ***if-then*** rules in backward or forward chaining [1] [3].

Evaluation: When the stage Development of a prototype of expert system is finished it is necessary the process of testing. The last should be done according to some criteria of efficiency. In addition, other extraneous experts are invited to test the expert system regarding its efficiency by means of diverse examples.

Fitting: At this stage the fitting of the expert system with other programming tools is done in environment, in which it will operate.

Maintenance: This is the stage when the system is heavily used by users.

It should be emphasized that the process of the development of expert systems is connected to:

- Combining efforts of several specialists (specialists in the domain, specialists in informatics) - this creates a lot of difficulties with different nature, e.g. achieving a common language and others.
- The computer specialist has the initiative – during the whole process, a lot of proposals and versions are generated; the specialist should be able to overview well enough the domain.

- Domain experts usually have difficulties to describe and structure their ideas in the manner appropriate for a formal description.

5. RESULTS, DISCUSSIONS, CONCLUSIONS

In general, knowledge based expert systems can solve complex problems. There is a need they to be analyzed well before being implemented. If there are critical drawbacks in this phase, the last should be improved. There is a strong belief that the social society should use such system to solve one of the main problems of social area domain - what is the level of quality of life?

6. ACKNOWLEDGEMENT

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Analysis of the Human Resources of the Food Subsectors Through Benchmarking

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***Abstract**: In this paper we will present a project "Enhancement of the competitiveness and restructuring of the food subsectors through benchmarking" (F.IND. CONSULTING)– Contract B2.32.01 /17.02.2012 under the European Territorial Cooperation Programme "Greece-Bulgaria 2007-2013". It is based on the benchmarking methodology, using an innovative approach to promote the sector's performance and competitiveness. Benchmarking is the continuous search for and adaptation of significantly better practices that lead to superior performance by investigating the performance and practices of other organizations. Benchmarking goes beyond comparisons with competitors to understanding the practices that lie behind the performance gaps.*
(Review article)

***Keywords:** human resources, food subsectors, equation of Becker.*

1. INTRODUCTION

In 2008 when Europe and indeed the entire developed world was hit by the global financial crisis which brought with it the credit squeeze, a fall in house prices and tumbling stock markets together produced a slump in consumer confidence, consumption and investment. The European food industry has not escaped the shock: economic growth has slowed sharply and unemployment has increased in many countries for the first time in several years [7]. Even though the effects of the financial crisis did not have as significant an impact on the agro-food industry, as it has in other industries such as the automotive and financial services industries, this sector is also impacted by this economic crisis. Indeed the food industry is now facing a fundamentally different "playing field" than it faced a decade ago. In this context, the achievement of the EU internal market and the evolution of EU food legislation are also relevant.

2. METHODOLOGY

The project is based on the benchmarking methodology, using an innovative approach to promote the sector's performance and competitiveness. Benchmarking is the continuous search for and adaptation of significantly better practices that lead to superior performance by investigating the performance and practices of other organisations. Benchmarking goes beyond comparisons with competitors to understanding the practices that lie behind the performance gaps. It is not a method for 'copying' the practices of competitors [7], but a way of seeking superior process performance by looking outside the industry.

In our work we used a mathematical model which is described from Kyurkchiev [3 - 6, 14, 15]

A company would be in equilibrium when its product (*MP*) is equal to the remuneration paid by the company (*W*), i.e.

$$MP = W$$

or for a given period *t*

$$MP_t = W_t \qquad (1)$$

Naturally, the disturbed equilibrium condition (1) between revenues and expenses can be offset by appropriate compensatory mechanisms in the next period t +1.

Thus, the equilibrium condition (1) will be transformed into

$$\sum_{t=0}^{n-1} \frac{R_t}{(1+i)^{t+1}} = \sum_{t=0}^{n-1} \frac{E_t}{(1+i)^{t+1}} \qquad (2)$$

where:

R_t – entire proceeds;

E_t – all costs;

n – number of periods [1, 2].

The costs for the remaining period will be equal to the salaries and incomes – the marginal product. Thus equation (2) yields the form:

$$MP_0 + \sum_{t=0}^{n-1} \frac{MP}{(1+i)^{t+1}} = W_0 + k + \sum_{t=0}^{n-1} \frac{W_t}{(1+i)^{t+1}} \qquad (3),$$

where with *k* are reflected the cost of funding the training program.

Easy to see that the equation of Becker can be written as follows

$$W_0 + k - MP_0 - \sum_{t=0}^{n-1} \frac{(MP_t - W_t)}{(1+i)^{t+1}} = 0. \qquad (4)$$

Obviously equation for financial management of human resources is an algebraic polynomial of degree *n* concerning the variable *q* where we have made such

$$q = \frac{1}{1+i},$$

i.e. is from the kind

$$C_0 - \sum_{p=0}^{n} C_p q^p = 0, \qquad (5)$$

In assessing the profitability of investments in human capital formation following are important variables characterizing the investment project:

$B_1, B_2, ..., B_n$ – flow of revenues

$C_1, C_2, ..., C_n$ – flow of costs;

$-I_0$ – front investment costs;

$B_1 - C_1, B_2 - C_2, ..., B_n - C_n$ – flow of net income;

r – the discount rate;

n – life of the project.

Net current value (*NVP*) is defined so

$$NVP = -I_0 + \frac{B_1 - C_1}{1+r} + \frac{B_2 - C_2}{(1+r)^2} + ... + \frac{B_n - C_n}{(1+r)^n}. \qquad (6)$$

Will explicitly noted that if $NVP \geq 0$, the investment project can be approved.

There is a direct proportional relationship between risk and return.

Internal form of return (*IRR*) is defined so

$$I_0 + \frac{C_1}{1+IRR} + ... + \frac{C_n}{(1+IRR)^n} = \frac{B_1}{1+IRR} + ... + \frac{B_n}{(1+IRR)^n}. \qquad (7)$$

IRR internal rate of return at which the net current value of the project is zero.

3. RESULTS

According to Nemuth at al. the theoretical deduction in the risk management circle and these phases show, that risks are in principle controllable and assessable [9].

The result of the Monte Carlo Simulation via Matlab is a probability distribution [8]. Figure 1 shows the probability density. Figure 2 shows the result under the cumulative ascending point of view [10-12].

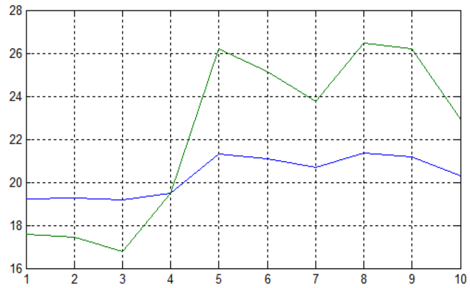

Fig. 1: Probability Density

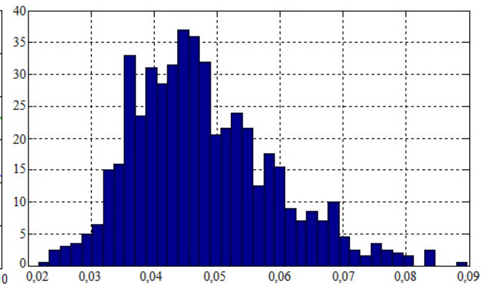

Fig. 2: Cumulative Ascending

Some results from our calculation are presented in Table 1. In future we will make an investigation in South West Bulgaria with firms from food subsectors.

Summary Statistics for Impact CO + Risk & Profit: / Risk Evaluation of Subcontractor Cost	
Minimum	3,48 %
Maximum	8,56 %
Mean	9,43 %
Median	7,02 %
Mode	5,98 %
Left X = VaR 5 %	6,90 %
Right X = VaR 95 %	7,95 %

Tab. 1: Summary statistics and results

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On Detecting Noun-Adjective Agreement Errors in Bulgarian Language Using GATE

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***Abstract**: In this article, we describe an approach for automatic detection of noun-adjective agreement errors in Bulgarian texts by explaining the necessary steps required to develop a simple Java-based language processing application. For this purpose, we use the GATE language processing framework [9], which is capable of analyzing texts in Bulgarian language and can be embedded in software applications, accessed through a set of Java APIs. In our example application we also demonstrate how to use the functionality of GATE to perform regular expressions over annotations for detecting agreement errors in simple noun phrases formed by two words – attributive adjective and a noun, where the attributive adjective precedes the noun. The provided code samples can also be used as a starting point for implementing natural language processing functionalities in software applications related to language processing tasks like detection, annotation and retrieval of word groups meeting a specific set of criteria.*

***Keywords:** Bulgarian grammar, Human language processing, GATE, NLP, POS tagger, JAPE transducer*

1. INTRODUCTION

The adjective in Bulgarian language must agree with the noun in number, gender, and definiteness. For example, in the noun phrase "щастливо деца" ("a happy children") the adjective and the noun do not agree in number – the adjective "щастливо" is singular and the noun "деца" is plural. In the noun phrase "щастлив дете" ("a happy child") the adjective and the noun do not agree in gender – the adjective "щастлив" is masculine and the noun "дете" is neuter. While these types of errors are rarely made by native speakers of Bulgarian, for people learning Bulgarian as a foreign language, this is not the case. This is why tools for automatic detection of such errors can be very useful.

Our choice to use GATE for this purpose is based mainly on the fact that it is a mature open source project with over 15 years of development in the field of human language processing [6] and provides extensive support for integration and interoperability with most of the other key systems and tools in this field [2].

Taking into account the specifics of the GATE framework, the task of automatic detection of noun-adjective agreement errors can be divided into a set of related subtasks:

- Breaking a text into sentences.
- Breaking a text into tokens (words, numbers, etc.).
- Tagging individual tokens with their respective part of speech (POS) tags.
- Tagging adjective–noun pairs that do not agree in gender, number and definiteness.

The first three subtasks are standard natural language processing (NLP) tasks – sentence splitting, tokenization and POS tagging [8]. The last subtask we will handle using the JAPE (Java Annotation Patterns Engine) transducer, which is part of the GATE system and provides finite state transduction over annotations based on regular expressions [1]. For more information about JAPE see [5].

Note that in this article we focus on the use of the GATE APIs for performing specific language processing tasks, which requires a basic understanding of the Java programming language. In addition, a basic understanding of regular expressions and writing JAPE rules is also required.

2. EXTERNAL LIBRARY DEPENDENCIES

To make the GATE APIs available to a Java based application, one need to deploy the following Java extensions:

- `$GATE_HOME/bin/gate.jar`
- all JAR files in `$GATE_HOME/lib`

Where `$GATE_HOME` is the absolute path of the GATE's root directory [1].

It is also mandatory to initialize the GATE library before first use. This is done by calling the static method `gate.Gate.init` [7].

3. LOADING CREOLE PLUGINS

Before using a certain processing resource in GATE, first the CREOLE plugin containing the resource should be loaded. In our example we will use four processing resources – LingPipe Tokenizer PR, LingPipe Sentence Splitter PR, LingPipe POS Tagger PR and JAPE-Plus Transducer. The first three resources are part of the LingPipe plugin. The last one is part of the

JAPE_Plus plugin. Our choice to use the LingPipe plugin is based on the fact that currently this is the only plugin, included in the GATE distribution, with POS tagger which includes models for Bulgarian language. We must, however, note that there is no restriction in using tokenizers and sentence splitters from another plugins in conjunction with the POS tagger from the LingPipe plugin.

To load the LingPipe and JAPE_Plus plugins, we need to register the corresponding CREOLE directories. This is done by issuing the following code [1]:

```
java.io.File f = new java.io.File (
    gate.Gate.getPluginsHome(), "LingPipe"
);
gate.Gate.getCreoleRegister().registerDirectories (
    f.toURI().toURL()
);

f = new java.io.File (
    gate.Gate.getPluginsHome(), "JAPE_Plus"
);
gate.Gate.getCreoleRegister().registerDirectories (
    f.toURI().toURL()
);
```

4. LOADING PROCESSING RESOURCES

The next step, after loading the plugins containing the aforementioned resources which will be used in our example, is to load the actual processing resources.

As pointed out in the introduction, we need processing resources for the following tasks:

- Breaking a text into sentences.
- Breaking a text into tokens.
- Tagging the individual tokens with their respective part of speech tags.
- Tagging adjective–noun pairs that do not agree in gender, number and definiteness.

The Sentence Splitter processing resource is used for breaking the text into sentences. To create it with default parameters, we use the following code:

```
gate.FeatureMap params =
    gate.Factory.newFeatureMap();

gate.ProcessingResource splitter =
```

```
        (gate.ProcessingResource)gate.Factory.createResource (
            "gate.lingpipe.SentenceSplitterPR", params
        );
```

To create a Tokenizer processing resource with default parameters, we use the following code:

```
    gate.FeatureMap params = gate.Factory.newFeatureMap();
    gate.ProcessingResource tokenizer =
       (gate.ProcessingResource)gate.Factory.createResource (
            "gate.lingpipe.TokenizerPR" , params
       );
```

Once the input text is broken into tokens, the LingPipe POS Tagger processing resource can be used for tagging the individual tokens with their respective part of speech tags. For this purpose, we need to provide a model for Bulgarian language as an initialization parameter [1, 7]:

```
    String model =
        "LingPipe/resources/models/bulgarian-full.model";
    java.io.File f =
        new java.io.File(gate.Gate.getPluginsHome() , model);

    gate.FeatureMap params = gate.Factory.newFeatureMap();
    params.put("modelFileUrl", f.toURI().toURL());

    gate.ProcessingResource tagger =
        (gate.ProcessingResource)gate.Factory.createResource (
             "gate.lingpipe.POSTaggerPR" , params
        );
```

Finally, we create a JAPE-Plus Transducer, which will be used to detect the adjective–noun pairs that do not agree in gender, number and definiteness. As an initialization parameter we need to provide the grammar file containing our custom JAPE rules:

```
    gate.FeatureMap params = gate.Factory.newFeatureMap();
    java.io.File f = new java.io.File("/jape/Example.jape");
    params.put("grammarURL", f.toURI().toURL());

    String transducerClass =
        gate.jape.functest.TransducerType.PLUS.getFqdnClass();

    gate.ProcessingResource japePlus =
        (gate.ProcessingResource)gate.Factory.createResource (
            transducerClass, params
        );
```

We will discuss the content of `Example.jape` file in Section 6.

5. CREATING AND RUNNING A CORPUS PIPELINE

One of the ways to run the already created processing resources in sequence is to use a *corpus pipeline*. For this purpose, we will use the class SerialAnalyserController [1]:

```
gate.creole.SerialAnalyserController controller;
controller = (gate.creole.SerialAnalyserController)
    gate.Factory.createResource (
        "gate.creole.SerialAnalyserController",
        gate.Factory.newFeatureMap(),
        gate.Factory.newFeatureMap()
    );
```

The next step is to add the already created processing resources to the controller:

```
controller.add(splitter);
controller.add(tokenizer);
controller.add(tagger);
controller.add(japePlus);
```

Note that the order in which the processing resources are added is important. For example, the POS tagger requires as an input the 'Token' and 'Sentence' annotations, which are created by the tokenizer and the sentence splitter. The JAPE transducer will require each 'Token' annotation to have a 'category' feature, set with the part of speech tag of the respective token, which is done by the POS Tagger. Note that, for example, by "'Token' annotation", we mean annotation of type 'Token'.

The controller that we use in our example accepts only processing resources that process documents, also called *language analysers*. Hence, the controller will require a *corpus*, which is basically a collection of documents. When executed, the controller runs every analyser over each document in the corpus. For our purposes we will create a corpus with a single document in it [7]:

```
gate.Corpus corpus;
corpus = gate.Factory.newCorpus("Test Corpus");
java.io.File f = new java.io.File("/texts/bg.txt");
gate.Document doc;
doc = gate.Factory.newDocument(f.toURI().toURL());
corpus.add(doc);
```

Next, we set the controller to use the already created corpus and execute the added processing resources in sequence.

```
controller.setCorpus(corpus);
controller.execute();
```

6. WRITING THE JAPE GRAMMAR RULES

In this section, as an example, we show how to create a JAPE grammar rule, which matches all adjective–noun pairs, where the adjective is in plural form and the noun is in singular form. The matched adjective–noun pairs are then annotated as type 'PSAgrError'. We will call this rule PluralSingularPair.

Note that it is essential to specify at the start of the JAPE grammar the input annotations, against which the rules will be matched. In our case, we only need to iterate through the annotations of type 'Token'. So, the following line should be included in the JAPE grammar file "Example.jape":

```
Input: Token
```

In our example, the parts of speech are annotated by the LingPipe POS tagger using the BulTreeBank tagset (BTB-TS) scheme [4] and are added to the document as 'category' features of the 'Token' annotations. According to the BTB-TS tagset specification, if the first character of a POS tag is 'A', the respective token is an adjective, if it is 'N', the respective token is a noun. For nouns, the fourth character of the POS tag determines the number of the noun – 's' for singular, 'p' for plural. For adjectives, it is the third character. Hence, for example, to match all annotations of type 'Token' with 'category' feature, whose value is a string that starts with the capital letter 'A', followed by an arbitrary character and the lower-case letter 'p', we use the following pattern:

```
{ Token.category =~ "^A.p" }
```

To match all annotations of type 'Token' with feature 'category', whose value is a string that starts with the capital letter 'N', followed by two arbitrary characters and the lower-case letter 's', we use the following pattern:

```
{ Token.category =~ "^N..s" }
```

Note that the regular expression operator "^" matches the beginning of the line and the operator "." matches any single character [3].

Using these two patterns, we create a rule, which matches all adjective–noun pairs, where the adjective is in plural form and the noun is in singular form:

```
Rule: PluralSingularPair
Priority: 20
(
    { Token.category =~ "^A.p" }
    { Token.category =~ "^N..s" }
): pair
-->
:pair.PSAgrError = { rule = "PluralSingularPair" }
```

Note that in the last line of the rule, the temporary label "pair" is renamed to "PSAgrError", thus creating a new annotation of type 'PSAgrError'. Rules for covering other types of agreement errors can be implemented in the same manner.

7. CONCLUSION

As shown above, developing a language processing application using GATE does not require a significant learning curve. The time and effort invested is negligible and well-worth the benefits of the derived functionality [2]. The ability to perform regular expressions over annotations using the full power of pattern matching provides a wide range of techniques for solving all kinds of real problems related to computer processing of human language. Additionally, the limitations imposed by the JAPE grammar can be overcome by invoking custom Java code inside the JAPE rules.

We conclude that the GATE framework with its multilingual support and language-independent components is a reasonable choice for developing a software for automatic detection and correction of a wide range of grammatical errors in Bulgarian texts, which will be of great help for people learning Bulgarian as a foreign language.

8. ACKNOWLEDGEMENT

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A Model for HP Folding Prediction using Variable Size of Lattice

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Abstract*: Using the HP model for protein folding prediction, there are different approaches about the lattices – face-to-face cubic lattice or face-to-center cubic lattice, about using backbone structure or including side chains, but in all these models the size of the lattice is preliminary chosen according to the length of the peptide chain . The major idea, implemented in my experiment, is not to use lattice cube with constant size, but using flexible constrain for spreading away, which can vary according to the percentage of failing, caused by lack of space, in the process of folding.*

Keywords: *HP folding, Variable size of lattice, Integer programming*

1. Introduction

The 3D structure of proteins is the major factor that determines their biological activity. The synthesis of new proteins and the crystallographic analysis of their 3D structure is very slow and very expensive process. If we can predict the 3D structure of many proteins, than only proteins with expected properties have to be synthesized. That will increase the number of known structures in the databases for proteins, and they can be used for drug design. The prediction of the 3D structure of proteins, if we know only the primary structure – the amino-acid sequence, is the protein folding problem. The reason for this process of folding in water environment is the interaction between water molecules and between amino-acids and water molecules. As water molecule has higher polarity than amino-acids, there is a minimum of energy when the protein is folded, not to spoil water to water interconnections [1, 2]. The way of folding is determined by the polarity or the hydrophobicity of different amino-acids, so the 3D structure with minimum energy is the real case. There is less energy when more hydrophobic (H) amino-acids are in contact in the core of the folded 3D structure and more polar (P) amino-acids are in contact with water. As we know the amino-acid sequence and the hydrophobicity of every amino-acid, we can predict the 3D structure – this method is called HP folding. It is needed to calculate the impact of all contacts between amino-acids, using chosen scoring function, for many possible 3D structures.

2. HP Folding in Lattice Method

HP folding in lattice is used for prediction of the 3D structure, using only the amino-acid sequence, in which:

- all amino-acids have to be placed in the vertexes of a lattice,

- neighbour amino-acids of the sequence have to be placed in neighbour vertexes in the lattice,

- only one amino-acid can be placed in any vertex,

There are different models of HP folding in lattice. The difference may be in the type of the lattice, in the size of the lattice, placing every amino-acid in single vertex as a backbone of the peptide or placing it in two neighbour vertexes for backbone chain plus side-chain radicals, in the scoring function and in the way of placing of the amino-acids [2, 3].

Major lattice types are FFCL (face to face cubic lattice) and FCCL (face to center cubic lattice). In FFCL every next amino-acid can be placed in 5 neighbour vertexes. In FCCL there are 11 neighbour vertexes, so the amino-acid chain can curve in more angles.

Size of lattice is usually chosen between square root of the length of sequence and third root of it.

In most of the models, the scoring function is the number of contacts (amino-acids placed in neighbour vertexes) between hydrophobic amino-acids, which are not neighbours in the sequence, where hydrophobic amino-acids are Alanine (A), Isoleucine (I), Leucine (L), Methionine (M), Phenylalanine (F), Proline (P), Tryptophan (W), Valin (V) [2, 4, 5].

After a great number of variants for placing the amino-acids had been built and their score function had been calculated, the variant with the biggest value of the scoring function is probably the real case.

Validation of a model of HP folding in lattice can be done by checking if the neighbour amino-acids in the predicted 3D structure are neighbours in the known 3D structure in the database.

3. A Model using Variable Size of Lattice

In this research, the presented model has the following features:

1) The lattice is FFCL and it has integer 3D coordinates for the vertexes, but the size is not constant – there is a variable constrain for placing every next amino-acid, which has being changed during the calculations, according to the percentage of failing to build a possible 3D structure, which is explained bellow. The first amino-acid is placed in the vertex with coordinates (0,0,0) and every next is being placed in neighbour

vertex, using random numbers from 0 to 5 to choose one of six possible placements. If the number is 0 – x=x+1, if it is 1 – x=x-1, if it is 2 – y=y+1, and so on to number 5, as only one dimension has been changed. There is a check and, if this vertex is not already occupied, we may place it there. The second check has the role of limit for the size of the lattice – if new vertex (x,y,z) is far away from (0,0,0) in any direction, another random number for new direction has been wanted, as if it is occupied. There is a counter of new attempts for every next amino-acid, and if this counter become bigger than a determined parameter, the current 3D structure fails to be build and the cycle continue with the next possible. How far an amino-acid can be put is limited by the following constrains, where s is a variable parameter, l is the length of the amino-acid sequence:

(1) $x^3<s.l$, $x^3>-s.l$, $y^3<s.l$, $y^3>-s.l$, $z^3<s.l$, $z^3>-s.l$

This parameter starts with value 2 and grows up with 1 in every 1000 calculated possible 3D structures if there are more than 500 failed ones. If the sequence is longer, s usually becomes bigger. The problem is that more space might be needed to build possible 3D structure, but if there is much space, more 3D structures with lower scoring function have being built. That is why the variable limit of size is implemented, using this parameter s.

2) The scoring function is calculated by the count of neighbour amino-acids, so using cycle, it is checked for every amino-acid, how much neighbour occupied positions it has. The value of the scoring function is the sum of all contacts. Actually there is no meaning if a contact (neighbourhood) of amino-acids may be calculated twice (while checking every one with every other) or if neighbourhood of two ones, which are neighbours in the sequence, because it increases the scoring function equally and cannot change the choice of the biggest value. The following operation is used in the program to increase the value of the scoring function, where f_new is the new value, f_old is the old value, h[i] and h[j] are values of hydrophobicity of the amino-acids (1 or 0) with positions i and j in the sequence, currently reached by the cycle of the program:
(2) f_new=f_old+h[i].h[j]

4. Results

This program has being developed for now to work with smaller proteins, peptides with up to 74 amino-acids have been successfully tested. Values of the scoring function using (2) with different values of parameters s and trying 1000000 times to build possible 3D structures for protein 1UUB with 57 amino-acids, sequence

DFCLEPPYTGPCRAAIIRYFYNAKAGLCQTFVYGGCRAKSNNFKSAEDCM RTCGGA, taken from FASTA format by RCSB PDB are presented in Table 2. (the optimal value of s for bigger proteins is more difficult to be determined without using variable s).

s	
variable	205
2	0
3	0
4	205
5	205
6	205
10	156
20	112
30	0

Table 2. Values of scoring function

5. Conclusion

This research will continue with many other proteins using different values of parameters. This approach, using non-integer scoring function and variable size of lattice, might be used for development of new models for HP folding, in addition to other improvements.

6. Acknowledgment

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Optimization of Homology modeling of the δ-opioid receptor by Molecular Operating Environment

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Abstract: *Opioid receptors are trans-membrane proteins and their structures are not fully defined. This problem could be solved by different virtual approaches, specific software and computational tools. Homology modeling is an approach of choice for building theoretical models. Molecular Operating Environment was used for generating the human δ-opioid receptor (DOR) model. The aim of the present study was to apply the algorithm steps of HM to DOR. This application allows us to safely use rapidly generated in silico protein, such as the DOR model and to use it in further study concerning its interactions with δ-opioid selective ligands.*

Keywords: *Homology modeling; Molecular Operating Environment; δ-opioid receptor.*

1. INTRODUCTION

Homology modeling (HM) is a computational technique, within structural biology, to determine the 3D structure of proteins [1]. It uses available, high-resolution protein structures to produce a model of a protein of similar, but unknown, structure [2].

The algorithm of the HM process used in this work is presented in Figure 1. HM seeks to predict the 3D structure of a protein based on its sequence similarity to one or more proteins of known structure. The method relies on the observation that the structural conformation of a protein is more highly conserved than its amino acid sequence. HM can be divided into four steps: template identification, alignment, model building refinement, and validation with various tools computational available for each step (Tab. 1). More detailed information is available in recently published reviews [3-8].

The opioid receptors are a very important class of receptors that attract the attention of a huge number of scientists. Development of effective and selective ligands to each opioid receptor is a time consuming process, which involves knowledge and skills of different researchers: including chemists, biologists, medics and pharmacologists. Large numbers of compounds with opioid action were synthesised, characterised and biologically tested, but only some of them have the desired efficacy and selectivity to the respective opioid receptor.

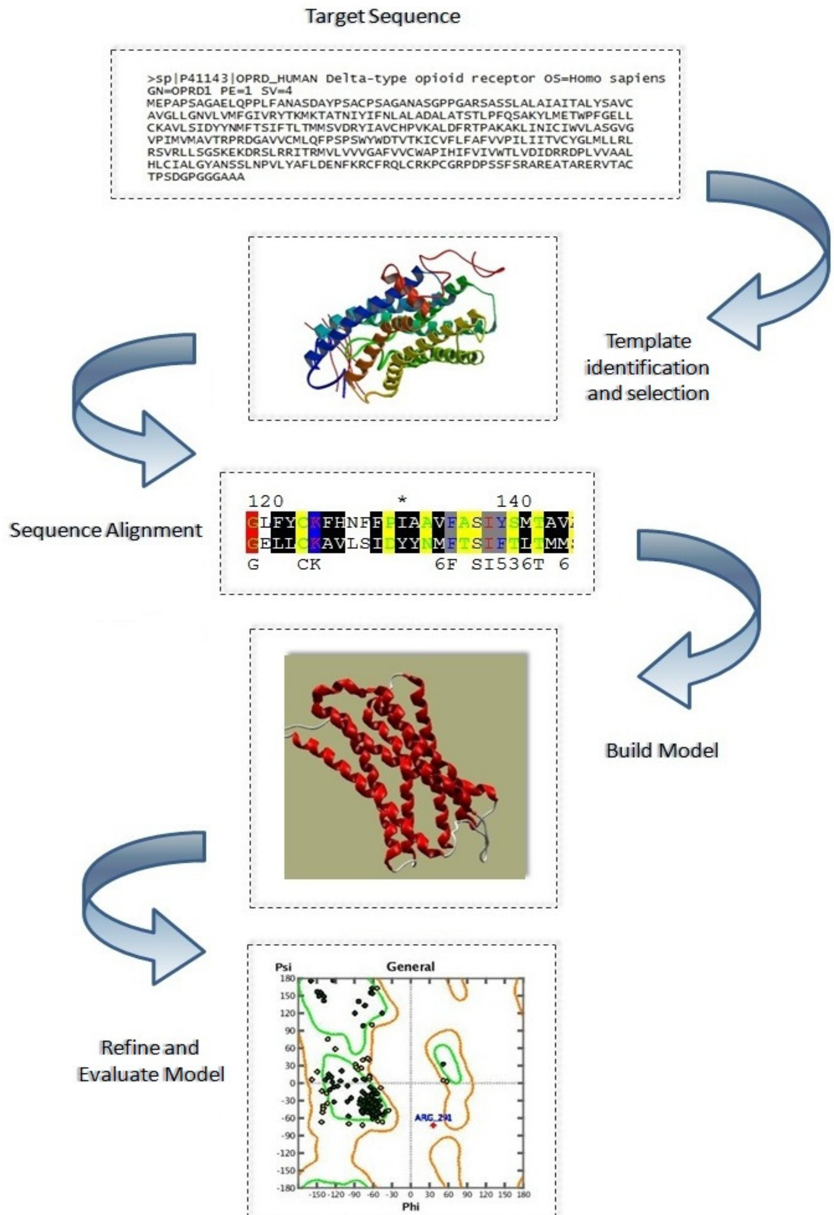


Fig. 1. A simplified illustration of the modeling process

On the other hand isolation and crystallisation of opioid receptors is an arduous task, due to their location. All of them are trans-membrane proteins and isolation causes destruction of their tertiary structure. So, these are the reasons to search for different approaches for solving this problem. Many experiments, concerning mutation of the binding site of opioid receptors, have been done in order to determine the key amino acid positions in the receptor, which are responsible for their action. Different theoretical models were proposed but they are not available for the investigators in that field, because theoretical models have not been published in the data base.

Tab. 1: Number of protein and protein/nucleic acid complex structures obtained by various experimental methods, available in the PDB as of 14 March 2013 (modified from www.rcsb.org/pdb/statistics/holdings.do).

Experimental method	Proteins	Nucleic Acids	Protein/NA complexes	Other	Total
X-RAY	73053	1442	3755	2	78252
NMR	8629	1022	192	7	9850
El. microscopy	349	41	124	0	514
HYBRID	46	3	2	1	52
Other	147	4	5	13	169
Total	82224	2512	4078	23	88837

The aim of our study is to build reliable model of the human δ-opioid receptor (DOR) with a help of Molecular Operating Environment (MOE).

2. MATERIALS AND METHODS

MOE feature is an advanced, comprehensive set of protein modeling applications that range from sequence analysis and alignment, multiple-structure alignment and superposition, remote homologue identification to 3D model building.

Chemical Computing Group Inc. (CCG) used MOE to create homology models in 1998. After many years development of this software, now it can make models with high quality and reliability. HM was performed according to presented scheme (Figure 1).

MOE SearchPDB searches for protein structures that are homologous to a query amino acid sequence. The search tool uses MOE's protein alignment capability to test the query sequence against each entry in the MOE homology databank. This databank is a library of alignments that were generated by clustering the unique, high-resolution chains in the Protein Data Bank, augmented by data from public domain sequence-only databases into families of related structures. The key technology used in this process was MOE-Align.

The extremely high Z-score guaranteed, in practical terms, that the hit was a true positive.

A high Z-score of 15 is considered very significant and the sequences can confidently be described as homologues. For Z-scores in the range 5–15, the proteins are probably homologous, though more distant relatives and there may be significant variations in their structures, although the structural core would probably be conserved between them. For lower Z-scores, there is less certainty in the nature of the relationship.

MOE-Align is a powerful and flexible application for multiple sequence and structure alignment of protein chains. Among its distinguishing features is its ability to accurately calculate affine gap penalties when aligning alignments. MOE-Align also possesses a structure-based alignment facility, which can even be employed when aligning a group of sequences for which only some have associated 3D data available.

MOE-Homology creates full-atom, energy-minimised 3D models of amino acid sequences from one or more template structures. The underlying methodology is based on a combination of the *segment-matching* procedure of Levitt [9] and an approach to the modeling of *indels* based on that of Fechteler *et al.* [10].

By default, MOE-Homology creates 10 models, each of which is generated by making a series of Boltzmann-weighted choices of sidechain rotamers and loop conformations from a set of protein fragments selected from the built-in library of high-resolution protein structures. Each of the candidate models can be saved in a molecular database for further analysis, while an average model is created and then submitted to a user-controlled level of potential energy minimisation. One can pick any of MOE's standard molecular mechanics forcefields which include two variants of the AMBER forcefield as well as MMFF94.

Statistical outliers are detected as follows:

- *Omega and Chi1:* Deviations from the typically observed *cis* or *trans* conformations under the assumption of a Gaussian mixture for the omega angle are highlighted.
- *Phi and Psi:* A 10-degree grid was placed on the 2D square [-pi,pi] X [-pi,pi]. Each grid square was marked with one of four annotations: CORE, ALLOWED, GENEROUS and OUTSIDE according to observed frequencies in high-resolution structures in the PDB. This annotated grid was then used to evaluate the phi and psi angles of a proposed model.
- *Bond Lengths and Angles:* Bond lengths and bond angles are compared to distributions observed within high-resolution entries in the PDB.

- *Bond Lengths and Angles:* Bond lengths and bond angles are compared to distributions observed within high-resolution entries in the PDB.

The structural similarity between the model and the real structure from the PDB is assessed from the root mean square deviation (RMSD) values [11]. The RMSD approach has its attraction in the physical significance that it provides. After the optimal alignment of the structures, the RMSD values are given by Eq. (1):

$$RMSD = \sqrt{\frac{\sum_{i=1}^{N} d_i^2}{N}},$$

where d_i is the distance between the *i*th atom of the two conformations and *N* is the number of such distances [13]. In general, an RMSD value, which is less than 3A°, implies a fairly good similarity between the structures. Our model has RMSD 0.5555 < 3A°, which means that the generated model is built correctly.

3. RESULTS AND DISCUSSION

Homology modeling starts with template identification. It lays the foundations by identifying appropriate homologue(s) of known protein structure, called template(s), which are sufficiently similar to the target sequence to be modelled. Using MOE a simple search was performed by submitting the sequence of DOR obtained from Swiss Prot database (accession number P41143, Figure 2). It was found that the solution conformation of substance P in water with NK1R (PDB id:2ks9) has high similarity with DOR sequence.

```
>sp|P41143|OPRD_HUMAN Delta-type opioid receptor OS=Homo sapiens
GN=OPRD1 PE=1 SV=4
MEPAPSAGAELQPPLFANASDAYPSACPSAGANASGPPGARSASSLALAIAITALYSAVC
AVGLLGNVLVMFGIVRYTKMKTATNIYIFNLALADALATSTLPFQSAKYLMETWPFGELL
CKAVLSIDYYNMFTSIFTLTMMSVDRYIAVCHPVKALDFRTPAKAKLINICIWVLASGVG
VPIMVMAVTRPRDGAVVCMLQFPSPSWYWDTVTKICVFLFAFVVPILIITVCYGLMLLRL
RSVRLLSGSKEKDRSLRRITRMVLVVVGAFVVCWAPIHIFVIVWTLVDIDRRDPLVVAAL
HLCIALGYANSSLNPVLYAFLDENFKRCFRQLCRKPCGRPDPSSFSRAREATARERVTAC
TPSDGPGGGAAA
```

Fig.2. Amino acid sequence of DOR.

The second step involves creating an alignment of the target sequence (DOR) with the template structure (Figure 3). The procedure was performed in MOE and the alignment is presented in Figure 3. It was found in sequence alignment that there was a sequence similarity greater than 50% in almost all transmembrane regions. Thus, it could be expected that homology models built with this alignment would be accurate.

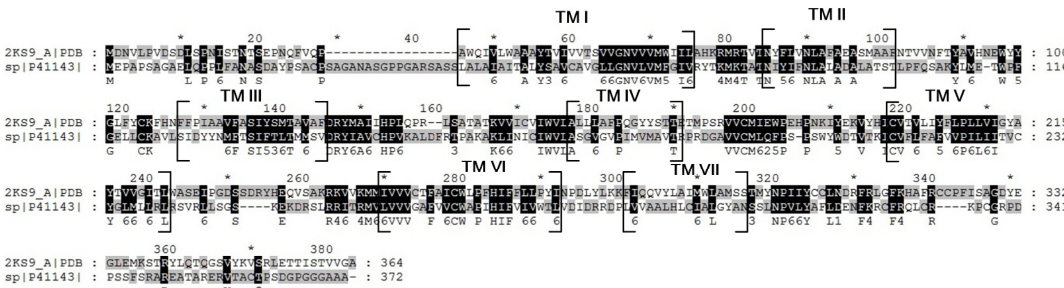
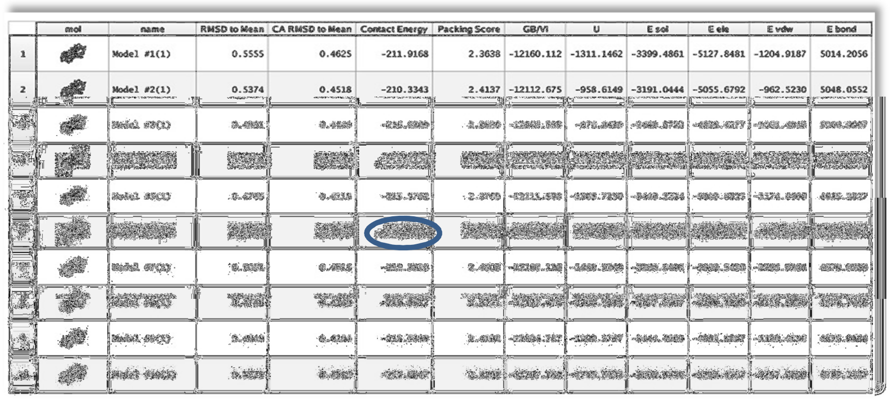


Fig.3. Sequence alignment of DOR with 2ks9 obtained from MOE and percentage of residue identity TM I – 100%, TM II – 87%, TM III – 83%, TM IV – 67 %, TM V – 73%, TM VI – 93%, TM VII – 33%: (color cod pattern: gray – substituted residues all templates; black – identical in both templates).

The next step is model building which used standard molecular mechanics forcefield AMBER 99. The algorithm within MOE consists of the following steps: (1) Initial partial geometry, where all coordinates are copied if residue identity is conserved; (2) Boltzmann-weighted randomised sampling, which consists of a data-collection step and a model-building step.



	mol	name	RMSD to Mean	CA RMSD to Mean	Contact Energy	Packing Score	GB/VI	U	E sol	E ele	E vdw	E bond
1		Model #1(1)	0.5555	0.4625	-211.9168	2.3638	-12160.112	-1311.1462	-3399.4861	-5127.8481	-1204.9187	5014.2056
2		Model #2(1)	0.5374	0.4518	-210.3343	2.4137	-12112.675	-958.6149	-3191.0444	-5055.6792	-962.5230	5048.0552
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Fig.4: Screenshot of database obtain after modeling by MOE.

During data collection, backbone fragments are collected from a high-resolution structural database, and alternative side-chain conformations for non-identical residues are assembled from an extensive rotamer library. During model building, ten independent models are created based upon loop and side-chain placements scored by a contact energy function (Figure 4).

The model with the best contact energy was chosen and it was validated directly in MOE. The stereochemical quality of the modeled proteins is assessed from Ramachandran validation score for favoured regions and allowed regions [14] (Figure 5). In general, a score close to 100% implies good stereochemical quality of the models.

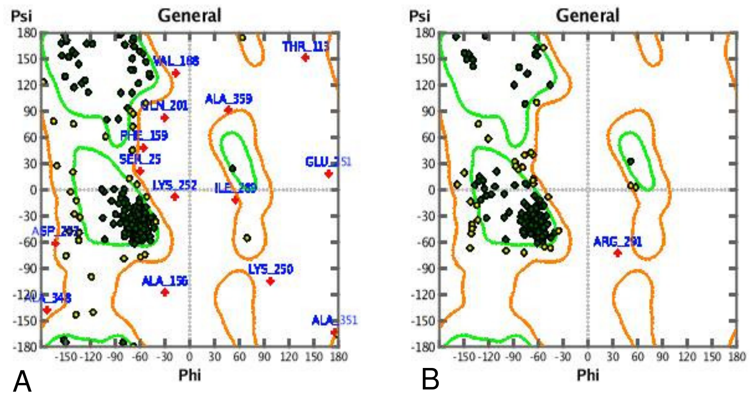


Fig.5: Ramachandran Plot generated by MOE: A-before optimization of the structure; B-after optimization of the structure.

In our case this score is 99.6 % and indicating a good stereochemical quality of our model.

The generated model of DOR obtained by HM could be used in further investigations. It could serve as a target in docking studies for design of new selective and effective DOR ligands.

4. CONCLUSION

A new model of DOR was generated and evaluated by MOE. According to our findings the model has good stereochemical quality, because residues are in a favoured position according to the tests and RMSD shows that the model is built correctly.

5. ACKNOWLEDGEMENTS

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Variable Neighborhood Search Based Algorithm for University Course Timetabling Problem

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***Abstract**: In this paper a variable neighborhood search approach as a method for solving combinatory optimization problems is presented. A variable neighborhood search based algorithm for solving the problem concerning the university course timetable design has been developed. This algorithm is used to solve the real problem regarding the university course timetable design. It is compared with other algorithms that are tested on the same sets of input data. The object and the methodology of study are presented. The main objectives of the experiment are formulated. The conditions for conducting the experiment are specified. The results are analyzed and appropriate conclusions are made. The future trends of work in this field are presented.*

***Keywords:** variable neighborhood search, university course timetabling problem.*

1. INTRODUCTION

In solving the combinational optimization problems usually several different algorithms are used that are tested on the same input data. For each specific problem and even for each different set of input data it is necessary to experiment and determine which algorithm and under what conditions (input parameters) would yield the best results for an acceptable time.

For solving the university course timetabling problem in [3] two algorithms are proposed, respectively genetic (GA) and mimetic (MA). The algorithms are tested on real data, obtaining good results. GA quickly finds an acceptable solution evaluation. In contrast, MA finds better solutions, but requires more computational time. Both algorithms are implemented in a real information system for automatic generation of a university course timetable, which is presented in [5]. Also, there is an available web service (presented in [4]), through which input data sets (used to compile a real university course timetable) can be downloaded. On these sets of input data different algorithms (not necessarily heuristics) can be tested that can be adapted and used to solve the studied problem.

In heuristic approaches it is typical that there is a space for many acceptable solutions [3]. Thus the problem is reduced to finding the optimal solution for a given criterion for optimality or a near optimal solution.

The focus in this paper falls on examining the approach VNS (Variable Neighborhood Search) and the development of an algorithm based on this approach. The algorithm must use the valuation model of university course timetable proposed in [2]. This will allow the algorithm to be integrated into the existing information system presented in [5] and to be compared with the already implemented algorithms in it (in the case of GA and MA).

2. THE VARIABLE NEIGHBORHOOD SEARCH APPROACH

Variable Neighborhood Search (VNS) is metaheuristic approach which aims to reduce the local minimum and greatly increase the local maximum by changing the adjacent elements when searching for solutions. This approach is often used in cluster analysis, theory of schedules, artificial intelligence and others.

VNS approach was first introduced in [7]. There are various modifications such as: Variable Neighborhood Descent (VND), Reduced VNS (RVNS), Basic VNS (BVNS), Skewed VNS (SVNS), General VNS (GVNS), VN Decomposition Search (VNDS), Parallel VNS (PVNS), Primal Dual VNS (P-D VNS), Reactive VNS, Backward-Forward VNS etc. [6].

The main idea of the VNS approach [1] is described below.

Consider the following optimization problem:

$$\min f(x) \tag{1}$$

$$\min x \in X, X \subseteq S \tag{2}$$

where $f(x)$ is the real objective function which should be minimized. The X is a set of feasible solutions, x is an acceptable solution and S is a space of solutions.

If S is a finite set, which is usually large, then combinatorial optimization problem is defined.

The solution $x^* = X$ is optimal if

$$f(x^*) \leq f(x), \forall x \in X \tag{3}$$

For the problem to be solved it is necessary to find an optimal solution x^*. If such an optimal solution doesn't exist then $X = \varnothing$.

When looking for solution, often it is allowed a tolerance, i.e. calculations stop if an acceptable solution x^* is found such that

(4) $$f(x^*) \le f(x) + \varepsilon, \forall x \in X$$

or

(5) $$\frac{f(x^*) - f(x)}{f(x^*)} \le \varepsilon, \forall x \in X$$

where ε is an acceptable tolerance value.

Sometimes an approximate solution is quicly found by using heuristics approaches, meaning that the resulting solution x_h is satisfactory:

(6) $$\frac{f(x_h) - f(x)}{f(x_h)} \le \varepsilon, \forall x \in X$$

for some ε, which is often large. But the optimality of the solution is not verified.

On the other hand, heuristic methods, which aim to avoid excessive computation time, often face another problem - finding a local optimum.

A local optimum of (1) – (2) is:

(7) $$f(x_L) \le f(x), \forall x \in N(x_L) \bigcap X$$

where $N(x_L)$ are neighborhood of x_L.

3. VARIABLE NEIGHBORHOOD SEARCH BASED ALGORITHM FOR UNIVERSITY COURSE TIMETABLING PROBLEM

The idea of the algorithm is as follows:

Step 1: Distribute N events $(n_1, n_2, ..., n_N)$ in k neighborhood structures $(NS_1, NS_2, ..., NS_k)$, so that $k \ge 2$ and $k \le N/2$. Consider the case in which there are at least two structures formed and in each structure there are at least two events, i.e. $|NS_i| \ge 2, i = 1,2,...,k$.

Step 2: Select the first structure, $i = 1$ (i.e. NS_1).

Step 3: Using a method based on local search to find a solution. If a solution is found, it is evaluated and its cost is stored in a temporary list. Furthermore, and the number (index) of the event is recorded, which is the first position in the structure. If the method based on local search does not find a solution, go to step 4.

Step 4: Rearrange the events in the current structure, shifted left (or right) with one position (see Fig. 1).

NS_{i-1}			NS_i					NS_{i+1}	
...	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}	...

NS_{i-1}			NS_i					NS_{i+1}	
...	n_8	n_9	n_{11}	n_{12}	n_{13}	n_{14}	n_{10}	n_{15}	...

Fig. 1: Shift events in i-th structure with one position left.

If any of the events has visited the first position in the structure (i.e., has turned a full circle), go to step 5, otherwise go to step 3.

Step 5: Rearrange the events in the current structure, with such shifts to the left or right that the first position is occupied by the event which had a solution with the best cost (according to the criteria used for optimality, it is solution in which the cost is lower). The list for temporary storage of costs and indices events is cleared.

Step 6: Choose the next NS_{i+1} structure, i.e. $(i = i + 1)$, and go to Step 3. If there are no more structures, i.e. $i > k$, then End.

If applied, the method based on the local search on the last order of events in the structures, and then the resulting cost will be the best for the so-formed structures. It is possible that this cost is achieved at an early stage of the algorithm, and then improvement is not found.

The pseudo-code of this algorithm is presented in Fig. 2.

```
begin
 N events are divided into k structures NS[][] //k ≥ 2, k ≤ N/2
 for each structure NS[i][] do //i := 1 to k
   for each event in current structure NS[i][j] do //j := 1 to |NSi|
     // beginning of the method based on local search
     for each event n do // n := 1 to N
       if event is fixed then go to next event //n := n + 1
       for each timeslot TS[t] do //t := 1 to T
         try to put the event n in the timeslot TS[t]
         if it is not possible go to next timeslot TS[t+1] // t := t + 1
         if the event is placed, calculate the cost of the solution:
         Cost(CurrentSalution), and store it: SaveCost(BestSalution),
         after that go to the next timeslot //t := t + 1
       end //for each timeslot
       put current event n in this timeslot TS[t], wherein the
       intermediate cost of the solution (BestSalutionCost) was the best,
       then go to the next event //n := n + 1
     end //for each event n
     // end of the method based on local search
     save the cost of finding solution (Solution) to a temporary list L
     L.cost.Add(Cost(Solution))
     save in L the index j of the event n, which is the first position in
     the current structure NS[i], L.index.Add(NSi[1].j))
     then go to the next event in NS[i][j] // j := j + 1
   end // for each event in current structure
```

```
   find the best cost among the saved in L
   best_cost := min(L.cost);
   best_index = L[best_cost].index;
   Rearrange the events in the current structure NS[i], so that the first
   position is the event, in which cost is the best
   for j := 1 to Length(NSi) do
   begin
     if (j = best_index) then break else NSi[j].index := NSi[j].index - 1;
     if NSi[j].index = 0 then NSi[j].index = NSi[Length(NSi)].index;
   end
   clear list for temporary storage of costs and indices Clear(L)
   then go to the next structure //NS[i+1]
end //for each structure
with this arrangement of events in structures, generate solution
by the method based on local search
end //of the algorithm
```

Fig. 2. Pseudo-code for generating a solution based on VNS approach.

In performing the method based on local search for each event n each time interval t is checked. The found solution is evaluated by function Cost (Solution), which has linear complexity $\Theta(N)$ depending on the number of events N. Since the method based on the local search iterated over all events N and placing each of them calls the function Cost (Solution), then the complexity of the entire process suggests that it requires $N.N$ actions i.e. complexity of the method based on local search is quadratic $\Theta(N^2)$.

In calculating the complexity of the entire algorithm one should take into account the fact that in the outer loop k number of times (as the number of formed structures) is iterated, and each iteration of this loop a nested loop is executed through which all elements in the current structure $|NS_i|$ are visited. That is after all the iterations of the outermost loop (for k) all events N will be passed exactly once (each event is assigned to exactly one of these k structures). At each step of the nested loops iterating through the events distributed in structures will call the method based on local search. It has a quadratic complexity as mentioned above. Therefore $N.N^2$ operations, i.e. N^3 are carried out. Then, operations are performed by storing values (costs of the solutions found and indices of events) that have $\Theta(1)$ complex. At the last stage, a search of the best cost (and index event) for each of the structures formed with subsequent rearrangement events in the respective structures is performed. These processes have the linear complexity $\Theta(Length(NS_1))$ depending on the number of elements in each structure. These operations do not affect significantly the execution time of the algorithm. Therefore, the complexity of the whole algorithm is cubic, i.e. $\Theta(N^3)$, dependent on the number of events N.

4. EXPERIMENTAL RESULTS

Three experiments with the following objectives were made:

1. To verify the efficiency of the prototype.
2. To determine the influence of the number of the formed neighborhood structures on the quality of the solution on different sets of input data.
3. To make a comparative analysis between the proposed algorithm and other algorithms used to solve the discussed problem. To compare the performance quality of the generated solutions and execution time of these algorithms. Different algorithms to be tested on the same sets of input data.

4.1. Improve the existing prototype

For the purpose of the experiment the prototype used for analysis of GA and MA described in [3] was improved. The algorithm, which was discussed in the previous section was implemented in the prototype and can be used for automated compilation of university course timetables. Using a prototype the planned experiments were made. From the results, the conclusions were made.

4.2. Conditions for the experiment

Experiments were conducted on a PC with 32 bit operating system Windows 7 Professional (Service Pack 1) and the following hardware configuration: Processor: Intel(R) Core(TM) 2 Duo CPU T7500 @ 2.20GHz 2.20GHz; RAM memory: 2.00 GB.

4.3. Methodology of the experiment

For the purposes of the experiment, 3 sets of input data were used, which are presented in [3]. Input data set "N18" with 18 events, 52 students (divided into 4 groups), 10 lecturers and 10 auditoriums (corresponding to 1 course). Input data set "N90" with 90 events, 175 students (divided in 14 groups), 29 lecturers and 18 auditoriums (corresponding to 1 subject). Input data set "N130" with 130 events, 274 students (divided in 21 groups), 37 lecturers and 22 auditoriums (set of 2 subjects with common events).

4.4. Results from the experiments conducted

Tab. 1 shows the parameters of GA and MA. The time of execution for a start of a reproduction is also shown.

Tab. 1: Parameters of GA and MA.

Parameters of GA and MA.	**N18**	**N90**	**N130**
Number of individuals in the population	32	32	32
Number of individuals crossing	16	16	8
Number of parental pairs	8	8	4
Number of received descendants	8	8	4
20% of descendants mutate	3	3	2
Number of reproductions (iterations)	50	50	50
Number of solutions to 1 iteration	43	43	38
Number of solutions for 1 start	2150	2150	1900
Execution time of the MA	0,31 sec.	11,25 сек.	27,68 sec.
Execution time of the GA	0,03 sec.	0,14 сек.	0,22 sec.

After conducting the experiments with GA and MA the results shown in Tab. 2 were obtained.

Tab. 2: Results of the GA and MA..

GA-N18		MA-N18		GA-N90		MA-N90		GA-N130		MA-N130	
#	Cost	#	Cost	#	Cost	#	Cost	#	Cost	#	Cost
1	24,85	1	1,19	1	89,92	1	13,49	1	151,53	1	19,72
2	25,51	2	1,16	2	89,00	2	15,16	2	117,75	2	21,16
3	24,90	3	1,27	3	88,36	3	13,21	3	114,90	3	22,77
4	24,09	4	1,36	4	91,34	4	12,44	4	117,36	**4**	**18,47**
5	26,05	5	1,25	5	89,93	5	15,03	5	116,72	5	20,85
6	25,00	6	1,19	6	87,59	**6**	**11,65**	6	111,95	6	20,32
7	23,44	**7**	**1,15**	7	87,71	7	12,28	7	113,01	7	19,12
8	23,37	8	1,18	**8**	**86,17**	8	12,29	8	113,30	8	20,64
9	**22,89**	9	1,21	9	87,97	9	11,93	**9**	**111,38**	9	21,43
10	24,34	10	1,24	10	89,33	10	13,52	10	117,15	10	19,02

The results shown are the performance of GA and MA in 10 starts for each of the input data sets. The input data set N18 (of 21500 total solutions for both algorithms) shows that the MA best solution is obtained from the seventh starting and has a cost of 1.15. For the GA the best solution is obtained from the ninth starting, which has a cost of 111.38. For input data N90 (also of total solutions 21500 for each of the two algorithms) shows that the MA best solution is obtained from the sixth starting, which has a cost of 11.65. For the GA the best solution is obtained at the eighth start, respectively having cost 86.17. The third input data set N130 (from a total number of 19000 solutions for each of the algorithms) shows that the MA best solution is obtained from the fourth starting, which has a cost of 18.47.

For the GA the best solution is obtained from the ninth starting, which has a cost of 111.38.

After conducting experiments with VNS-based algorithm, tested on input data set N18, the results of it are presented in Tab. 3. The execution time for *k* = 2 is 0.03 sec.

Tab. 3: Results of VNS-based algorithm for input data set N18 (a) k = 2, b) k = 3, c) k=6).

k = 2, |NSk| = 9			
NS1	Cost	NS2	Cost
1	2,85	10	2,16
2	2,87	11	1,85
3	2,85	**12**	**1,85**
4	2,85	13	5,91
5	2,85	14	3,74
6	3,61	15	5,16
7	3,61	16	5,14
8	2,18	17	2,90
9	**2,16**	18	2,85

a)

k = 3, |NSk| = 6					
NS1	Cost	NS2	Cost	NS3	Cost
1	2,85	7	2,84	**13**	**2,16**
2	2,86	8	2,18	14	3,58
3	4,30	9	2,17	15	3,63
4	3,56	10	2,17	16	2,87
5	2,84	11	2,16	17	2,87
6	**2,84**	**12**	**2,16**	18	2,91

b)

k = 6, |NSk| = 3											
NS1	Cost	NS2	Cost	NS3	Cost	NS4	Cost	NS5	Cost	NS6	Cost
1	2,85	4	2,84	**7**	**2,84**	10	2,85	13	2,84	16	2,84
2	2,85	5	2,84	8	4,27	11	2,84	14	2,84	17	2,84
3	**2,85**	**6**	**2,84**	9	4,27	**12**	**2,84**	**15**	**2,84**	**18**	**2,84**

c)

From Tab. 3 it is seen that for input data set N18, the events are first divided into two structures. The best solution is found for the rearrangement of the events in the first structure when the first position in the structure was an event with index 9. The found solution has cost of 2.16. Then the algorithm is continued searching by rearranging the events in the second structure, the best solution was found when the first position in the second structure has event with index 12. In this arrangement the found solution has cost of 1.85. This is the best solution found so it formed neighborhood structures of events.

In the distribution of events in 3 structures the following results were obtained: at NS1 algorithm has found a solution with the best cost 2.84 when the first position in this structure was event with index 6. Note that this cost is obtained when in the first position in the structure was an event with index 5. In the second structure NS2, the algorithm finds the best solution with cost 2.16 when the first position in this structure was events with indexes 11 and 12 respectively. In rearranging the events of the third structure NS3 there is not an improvement.

In the distribution of events in the 6 structures, the best solution found has a cost of 2.84. It is obtained by rearranging the events in the second structure - NS2. Then no improvement is achieved.

For the input data set N90 were obtained results presented in Tab. 4. Execution time for $k = 2$ is 0.47 sec.

. Tab. 4: Results of VNS-based algorithm for input data set N90.

k	|NSk|	NSi	# - Index of the event, which is the first position in the structure. C – Solution cost.										
2	45	1..2	#	12	**58**								
			C	15,37	**14,75**								
3	30	1..3	#	12	41	**67**							
			C	15,37	14,64	**14,64**							
5	18	1..5	#	12	19	44	72	**75**					
			C	14,30	14,30	12,53	12,53	**12,39**					
6	15	1..6	#	12	30	33	60	67	**77**				
			C	12,78	12,78	12,78	12,25	12,25	**12,18**				
9	10	1..9	#	4	12	25	36	44	52	67	71	**88**	
			C	15,52	14,62	14,37	13,82	13,27	12,96	12,96	12,96	**12,68**	
10	9	1..10	#	4	11	19	28	45	46	55	70	70	**85**
			C	16,57	15,49	15,49	15,49	15,49	15,49	15,49	14,53	13,03	**13,03**
15	6	1..10	#	2	12	17	19	25	33	39	48	50	58
			C	17,41	16,59	14,99	14,99	14,99	14,99	14,69	14,41	14,41	14,41
		11..15	#	65	70	78	80	**85**					
			C	14,17	13,31	13,31	13,31	**13,31**					
18	5	1..10	#	3	9	15	20	21	26	33	36	45	50
			C	18,95	16,19	16,19	16,19	16,19	16,19	16,19	16,19	16,19	16,19
		11..18	#	52	60	65	67	75	76	85	**89**		
			C	16,19	16,19	16,19	16,19	16,19	16,19	16,19	**16,19**		

Tab. 4 shows that for an input data set N90, the events were divided into 2, 3, 5, 6, 9, 10, 15 and 18 structures. The best found solution is obtained when the events were divided into 6 structures (with 15 events in each structure). In rearranging the events in the last structure NS6, when the first position was an event with index 77, the generated solution has cost of 12.18. Note that the distribution of events in 5 and 9 structures has led to solutions close in cost to the best found, respectively, with costs: at 5 structures: 12,39 and at 9 structures: 12.68.

For the input data set N130 the results presented in Tab. 5 were obtained. Execution time for $k = 2$ is 0.89 sec.

Tab. 5: Results of VNS-based algorithm for input data set N130.

<table>
<tr><th>k</th><th>|NS|</th><th>NSi</th><th colspan="11"># - Index of the event, which is the first position in the structure.
C – Solution cost.</th></tr>
<tr><td rowspan="2">2</td><td rowspan="2">65</td><td rowspan="2">1..2</td><td>#</td><td>64</td><td>67</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>C</td><td>21,01</td><td>21,01</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">5</td><td rowspan="2">26</td><td rowspan="2">1..5</td><td>#</td><td>17</td><td>49</td><td>71</td><td>100</td><td>127</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>C</td><td>22,11</td><td>20,62</td><td>18,59</td><td>18,05</td><td>17,74</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">10</td><td rowspan="2">13</td><td rowspan="2">1..10</td><td>#</td><td>12</td><td>14</td><td>27</td><td>45</td><td>57</td><td>75</td><td>91</td><td>95</td><td>112</td><td>130</td></tr>
<tr><td>C</td><td>19,80</td><td>19,80</td><td>19,80</td><td>19,26</td><td>18,51</td><td>18,40</td><td>18,40</td><td>17,47</td><td>17,21</td><td>17,21</td></tr>
<tr><td rowspan="4">13</td><td rowspan="4">10</td><td rowspan="2">1..10</td><td>#</td><td>6</td><td>20</td><td>28</td><td>39</td><td>43</td><td>51</td><td>67</td><td>73</td><td>83</td><td>95</td></tr>
<tr><td>C</td><td>22,44</td><td>22,44</td><td>20,61</td><td>18,80</td><td>18,80</td><td>18,80</td><td>18,80</td><td>17,87</td><td>17,80</td><td>16,86</td></tr>
<tr><td rowspan="2">11..13</td><td>#</td><td>101</td><td>112</td><td>130</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>C</td><td>16,86</td><td>16,86</td><td>16,86</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="6">26</td><td rowspan="6">5</td><td rowspan="2">1..10</td><td>#</td><td>3</td><td>9</td><td>14</td><td>20</td><td>21</td><td>26</td><td>34</td><td>40</td><td>44</td><td>48</td></tr>
<tr><td>C</td><td>23,99</td><td>21,63</td><td>20,81</td><td>20,81</td><td>20,81</td><td>20,81</td><td>20,81</td><td>20,64</td><td>20,64</td><td>20,64</td></tr>
<tr><td rowspan="2">11..20</td><td>#</td><td>55</td><td>57</td><td>64</td><td>67</td><td>75</td><td>76</td><td>85</td><td>90</td><td>94</td><td>97</td></tr>
<tr><td>C</td><td>20,14</td><td>19,92</td><td>19,77</td><td>19,77</td><td>18,84</td><td>18,84</td><td>18,84</td><td>18,84</td><td>18,84</td><td>18,84</td></tr>
<tr><td rowspan="2">21..26</td><td>#</td><td>105</td><td>110</td><td>115</td><td>120</td><td>125</td><td>127</td><td></td><td></td><td></td><td></td></tr>
<tr><td>C</td><td>18,84</td><td>18,84</td><td>18,84</td><td>18,84</td><td>18,84</td><td>18,84</td><td></td><td></td><td></td><td></td></tr>
</table>

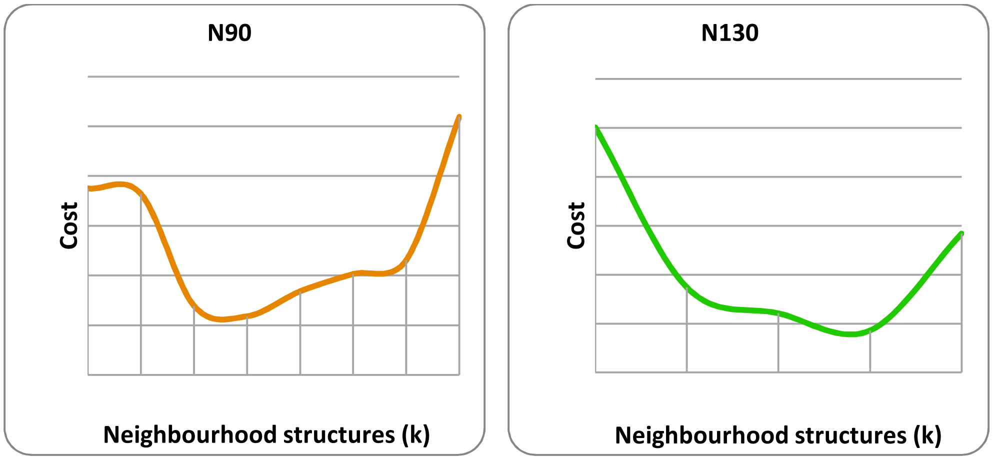
Tab. 5 shows that for an input data set N130, the events were divided into 2, 5, 10, 13 and 26 structures. The best found solution is obtained when the events were divided into 10 structures (with 13 events in each structure). In rearranging the events in structure NS10, when the first position was an event with index 95, the generated solution has cost of 16.86. This cost has not been improved in the next structures. Note that the distribution of events in 5 and 10 structures has led to solutions close in cost to the best found, respectively, with costs: at 5 structures: 17,74 and at 10 structures: 17.21.

4.5. Conclusions from the experiments

After conducting the experiments the following conclusions were made:

1. The developed algorithm can be used in solving the university course timetabling problem.

2. The number of formed neighborhood structures of events, affect the quality of the solution for all sets of input data. Fig. 3a and 3b show graphs of the input data set N90 and N130.



a) b)

Fig. 3: Behavior of VNS-based algorithm for different number of formed neighborhood structures of events, for the input data set N90 - a) and for N130 - b).

The graph shows that a small number of structures with more events and a large number of structures with few events get the worst results among all the results obtained. From the obtained experimental results, it can be concluded that for N90 best solutions are obtained when each of the formed structures is between 8,1% and 16,2% of the total number of events (in this case 90 events). For N130 best solutions are obtained when each of the formed structures is between 13% and 16.9% of the total number of events (in this case 130).

3. Following the experiment the results obtained in [3] are confirmed. It can be seen that MA gives much better results than GA for all sets of input data, but it requires more CPU time. VNS-based algorithm generates solutions that are much better than those obtained from DA and are commensurate with the solutions obtained from MA. For input data set N130 a solution is found which has a cost of 16.86, which is better than the best solution found by the MA, respectively, with a cost of 18.47.

By increasing the number of events execution time of GA and VNS-based algorithm increases slightly. In contrast, MA requires much more processor time to perform in large input data sets (see Fig. 4).

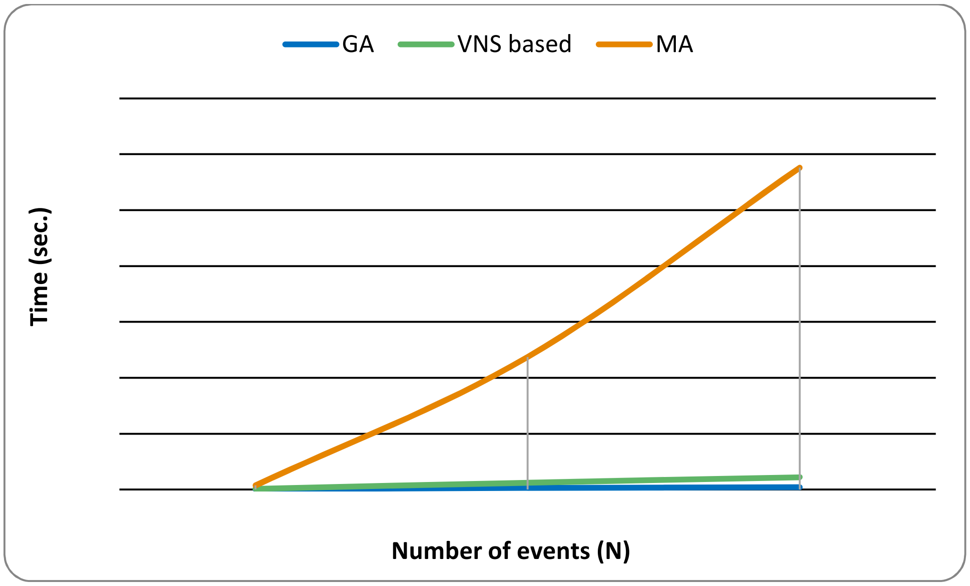


Fig. 4: Execution time of algorithms with increasing number of events.

5. CONCLUSION

In this article VNS based algorithm is presented. It was successfully used to solve the university course timetabling problem. The methodology and the object of the study are presented. Also, the objectives of the planned experiments are formulated. The proposed algorithm is compared with two other algorithms - GA and MA. All algorithms are tested on the same input data sets.

In conclusion, we note that the developed VNS based algorithm, can be used successfully in solving the university course timetabling problem as well as in small input data sets and large ones. By conducting experiments it is seen that for small input data sets, MA gives better results in a reasonable time. By increasing the number of events, however, the execution time of MA is much greater than the time for VNS-based algorithm. Furthermore, for the same input data sets it finds comparable solutions, and in some cases better than those found by the MA.

As future trends of employment seeking ways to optimize the algorithm in terms of execution time may be noted. It is also necessary to do a large-scale research to determine the optimum ratio between the number of the formed neighborhood structures, and the number of events in each structure.

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Prime Numbers in the Subsets of a Set

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Abstract: *Two sets with n and n^2 elements, have respectively $\binom{n}{2}$ and $\binom{n^2}{2}$ subsets with two elements each. Since n(n+1) $\binom{n}{2} = \binom{n^2}{2}$, the following question naturally follows: For which natural number n, in each set with n^2 elements exist n(n+1) subsets with n elements each, so that they have every subset with two elements of the given set repeated only once? The following question naturally follows: For which natural number n, in each set with n^k elements exist $n^{k-1}(n^{k-1} + \cdots + n + 1)$ subsets with n elements each, so that they have every subset with two elements of the given set repeated only once? The paper shows that, this is possible both in the particular and the general case, only when n is prime number.*

Keywords: *Set, subset, prime number*

This paper has been inspired by a national lottery game in Greece in which the player has to find 6 of 49 natyral number $1, 2, \ldots, 49$.

We are first going to present 56 subsets of the set $A = \{1, 2, \ldots, 49\}$, each of them composed of 7 elements carrying an interesing feature. In fact, each of these 56 subsets of the set A are composed of subsets each made of two elements of the set A, which occurs only once. Theoretically this is possible. Indeed, one set with seven elements has $\binom{7}{2} = 21$ subsets with two elements; as a result, these 56 sets will make $56 \cdot 21 = 1176$ subsets with two elements. On the othes side, the number of all subsets with two elements in the set A is equal to $\binom{49}{2} = 1176$.

There arises a questions: Is it possible to find for each natural number n, $n(n+1)$ subsets of the set $A = \{1, 2, \ldots, n^2\}$ with n elements each, so that they include all the subsets with two elements of the set A only once?

Theoretically this is possible. In practice, a subset with n elements of the set A has $\binom{n}{2} = \frac{n(n-1)}{2}$ subsets with two elements each, so that $n(n+1)$ subsets with n elements each will include $\frac{n^2(n^2-1)}{2}$ subset with two elements of the set A. On the other hand, in the set A there will be $\frac{n^2(n^2-1)}{2}$ subsets with two elements each. The followin theorem shows that such a thing is possible only when the natural number n is prime number.

Theorem 1. *Given the set $B=\{1,2,\dots,n^2\}$. Necessary and sufficient condition to make it possible that there should exist $n(n+1)$ subsets of the set B with n elements each, so that they include all of the subsets with two elements of the set B, is that the number n is prime.*

Proof. We first need to clarify what a full square is. In this case, it is a square which includes n^2 numbers of the set B. To prove this theorem we will need $n+1$ full squares. In order to find these $n(n+1)$ subsets of the set B, we will proceed as follows.

The first full square will include the n^2 numbers in their natural order $1,2,\dots,n$ in its first row, $n+1,n+2,\dots,2n$ in the second row, and in the n-th row numbers $n(n-1)+1,n(n-1)+2,\dots,n^2$ The second full square will be like the first one, but only if rotated in 90^0 around its center (the rows of the full second square will be the columns of the first full square and vice versa). While for the rest of the $n-1$ full squares, will be filled in the following way: the numbers that are positioned in the diagonal line of the full square with the number i, will constitute the first row of the full square with the number $i+1$, in which $i=1,2,\dots,n$. Their columns will have the same numbers in the same order (in their circular meaning). It can be noticed that the diagonal line of the last full square will be the first row of the full square with the number 2. It is clear that the full squares with the numbers $2,3,\dots,n+1$, are repeated periodically because of the above procedure.

The $n(n+1)$ subsets of the set B will altogether have $\frac{n^2(n^2-1)}{2}$ subsets each carrying two elements of the set *B*; therefore, in order to show that they include only once all of the subsets of the set B with two elements, it will be enough to prove that there does not exist any pair, which is simultaneously part of these $n(n+1)$ subsets.

Firstly, if two numbers occur in one of the rows of the first full square, they will not occur together in any of the rows of the following full squares, because in these later ones they will be parts of the same column. This is the reason why two numbers part of one of the rows in the full squares with numbers $2,3,\dots,n$ will not occur together in the other rows of the full squares with the same numbers.

In the first columns of the full squares with the numbers $2,3,\dots,n+1$, there are the numbers $1,2,\dots,n$; in the second columns of the same squares there are the numbers $n+1,n+2,\dots,2n$ and so on; in their last columns there will be the numbers $n(n-1)+1,n(n-1)+2,\dots,n^2$. It will be noticed that in the juxtaposed full squares the first columns remains unchanged, the second column moves with one unit upward (in a circular meaning), the third column moves with two units upward, and so on, up to the last column which moves with $n-1$ units upward (when a number which moves together with the columns, occurs in the first row of the full square, it will reach the last row of the full square during the entire movement).

Now let us consider the numbers x and y, which are in one of the rows of the full squares with the numbers $2, 3, \dots, n+1$. Let us suppose that the number x is in the column with the number i and the number y occurs in the column with the number l, where $i, l \in \{1, 2, \dots, n\}, i < l$. In the juxtaposed full squares the number x will move with $i-1$ units upward, while the number y will move with $l-1$ units upward. After m full squares which follow the full square including the numbers x and y, in which $m = 1, 2, \dots, n-1$, the difference of the rows in which these numbers will be will be equal to:

$m(l-1) - m(i-1) = m(l-i)$.

Yet, in order to have both numbers x and y in the same row of another full square, the number $m(l-i)$ must be a multiple of the number n.

Therefore, in case number n is prime, since $l-i < n$ and $< n$, we can conclude that the number $m(l-i)$ is not a multiple of the number n, which in its turn shows that the numbers x and y may in no way occur in the same row of the following squares. In this way we have proved that there is no repetition of the subsets with two elements of the B set each, by satisfying as a result the terms of the theorem. On the other hand, if it is possible to find the $n(n+1)$ subsets of the set B with n elements each, which include all of the subsets with two elements of the set B occurring only once, they can all be placed in $n+1$ full squares according to the procedure described above. This is possible because a random solution of the theorem (i.e. in $n(n+1)$ subsets with n elements each), we can only change the place of each two elements without breaking any of the theorem terms. In this way, if n is a compose number, then we would have $n|m(l-i)$ for the defined values of the numbers $m, l,$ and i (for example, if $n = ab$, for $m = a$ and $l - i = b$ we would have $n|m(l-i)$), which in its turn reveals the fact that there would be repetition of the determined pairs (the numbers x and y), and would be in contrast with the previous supposition). ■

Let us see now what happens in the set $C = \{1, 2, \dots, n^k\}, k \geq 2$. The question which arises is still the same. Is it possible to find $n^{k-1}(n^{k-1} + \cdots + n + 1)$ subsets of the set C with n elements each, for every natural number $n, k \geq 2$, so that they include all of the subsets with two elements of the set C occurring only once?

Theoretically this is possible. In practice, a subset with n elements of the set C has $\frac{n(n-1)}{2}$ subsets with two elements each, so that $n^{k-1}(n^{k-1} + \cdots + n + 1)$ subsets with n elements each will include $\frac{n^k(n^k-1)}{2}$ subset with two elements of the set C. On the other hand, in the set C there will be $\frac{n^k(n^k-1)}{2}$ subsets with two elements each. The followin theorem shows that such a thing is possible only when the natural number n is prime number.

Theorem 2. *Given the set $C = \{1, 2, \dots, n^k\}, k \geq 2$. Necessary and sufficient condition to make it possible that there should exist $n^{k-1}(n^{k-1} + \dots + n + 1)$ subsets of the set C with n elements each, so that they include all of the subsets with two elements of the set C, is that the number n is prime.*

Prove. The solution of this problem will be reflected through the mathematical induction of the number k. The argument of the theorem for $k = 2$ has been presented in the theorem 1. Let us suppose that the theorem is proved for the number $k - 1$ and let us try to prove it even for the natural number k.

A *full rectangle* is a rectangle sized $n^{k-1} \times n$, in which the elements of the set C occur only once. A *partial diagonal line* of a full rectangle is a diagonal line of the upper part of the rectangle, which forms a square.

First, there is need to form n^{k-1} full rectangles, therefore $n^{2(k-1)}$ subsets of set C as described below: In the first full rectangles the numbers of the set C will be placed in the following way; the numbers $1, 2, \dots, n^{k-1}$ will be placed in the first column downward, in the second column the numbers $n^{k-1} + 1, n^{k-1} + 2, \dots, 2n^{k-1}$ will still be in a downward position, and so on, up to the column n with the numbers $n^{k-1}(n-1) + 1, n^{k-1}(n-1) + 2, \dots, n^k$. For the rest of the $n^{k-1} - 1$ rectangles, the numbers will be placed in the partial diagonal line of the full rectangular with the number i, will constitute the first row of the full rectangle with the number $i + 1$, in which $i = 2, 3, \dots, n$; in the meantime, their columns will have the same numbers in the same order as in the first full rectangle (in their circular meaning). The partial diagonal line of the last full rectangle will constitute the first row of the first full rectangle. In other words, n^{k-1} full rectangles will be repeated periodically because of the above procedure.

For the rest, that is to say for $n^{k-2}(n^{k-2} + \dots + 1)$ subsets of the set C with n elements each, we will proceed as follows: With the elements of each set

$$1, 2, \dots, n^{k-1},$$
$$n^{k-1} + 1, n^{k-1} + 2, \dots, 2n^{k-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$n^{k-1}(n-1) + 1, n^{k-1}(n-1) + 2, \dots, n^k,$$

that is to say with the elements of each of the columns of the above full rectangles, we will form $n^{k-1}(n^{k-2} + \dots + 1)$ subsets with n elements each, so that they include all of the subsets with two elements of the respective sets. This is possible because of the supposition we have derived from the mathematical induction.

Let us now try to prove that $n^{k-1}(n^{k-1} + \dots + n + 1)$ subsets of the set C formed as above, with n elements each, prove the conditions of the theorem.

It can be noticed that in the juxtapositional full rectangles the first column remains unchanged, the second column moves with one unit upward (in a circular sense), the third column moves with two units upward, and so on up the last column, which moves with $n^{k-1}-1$ units upward (a number which moves together with the column, is to be found in the first row of the full square; during the movement it will reach the last row of the full square).

Now, lets the numbers x and y occur in the rows of the full rectangles with numbers $1, 2, \dots, n^{k-1}$. Let us suppose that the number x is placed in the column with the number i, and that the number y occurs in the column with the number l, in which $i, l \in \{1, 2, \dots, n^{k-1}\}, i < l$. In the juxtapositional full rectangles the number x will move with $i-1$ units upward and the number y will move with $l-1$ units upward. After m full squares which follow the full rectangle with the numbers x and y, in which $m = 1, 2, \dots, n^{k-1}-1$, the rest of the rows in which these numbers occur will equal:

$m(l-1) - m(i-1) = m(l-i)$.

However, in order to have the numbers x and y in the same row of another full rectangular, the number $m(l-i)$ must be a multiple of the number n^{k-1}.

Therefore, in case number n is a prime, since $l-i < n^{k-1}$ and $< n$, we can conclude that the number $m(l-i)$ is not a multiple of the number n^{k-1}, which in its turn shows that the numbers x and x may in no way occur in the same row of the following full rectangles. As a result the subsets with two elements each of the set C formed by the n^{k-1} full rectangles together with $n^{k-1}(n^{k-2}+\dots+1)$ subsets with the n elements part of the sets

$$1, 2, \dots, n^{k-1},$$
$$n^{k-1}+1, n^{k-1}+2, \dots, 2n^{k-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$n^{k-1}(n-1)+1, n^{k-1}(n-1)+2, \dots, n^k,$$

constitute the $n^{k-1}(n^{k-1}+\dots+1)$ subsets required in the set C. On the other hand, if it is possible to find the $n^{k-1}(n^{k-1}+\dots+1)$ subsets of the set C with n elements each, which include all of the subsets with two elements of the set C occurring only once, they can all be placed in $n^{2(k-1)}$ full rectangles; the rest will be placed according to the procedure described above (by making use of the argument based in the induction $k-1$). This is possible because a random solution of the theorem (i.e. in $n^{k-1}(n^{k-1}+\dots+1)$ subsets with n elements each), we can only change the place of each two elements without breaking any of the theorem terms. In this way, if n is a compose number, then we would have $n^{k-1}|m(l-i)$ for the defined values of the numbers $m, l,$ and i (for example, if $n^{k-1} = ab$, for $m = a$ and $l-i = b$ we would have $n^{k-1}|m(l-i)$), which in its turn reveals the fact that there would be repetition of the determined pairs (the numbers x and y) in the full rectangles, and would be in contrast with the previous supposition. As a result, we can conclude that n is prime number. ■

Impacts of Moodle on electrical engineering courses: opportunities and challenges

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Abstract: *Paper explores the influence of Moodle learning platform on the learning process of electrical engineering courses at University 'Goce Delcev' and makes the comparison between obtained results at several courses when Moodle was implemented as learning tool and not. The paper analysis based on survey results and data processing by statistical software package SPSS17, proved that transformation of teaching methods at high education institutions by implementing e-learning methods is going into right direction resulting in improved quality of learning and student satisfaction.*

Keywords: *e-learning, Moodle, University 'Goce Delcev'*

1. Introduction

During recent years, e-learning platforms as a part of Learning Management System (LMS) are becoming increasingly sophisticated by showing potential as an effective way of improving the learning process. Conducted researches in the last quarter of 2010 indicate that LMS market has settled around five products: Moodle, Sakai, Blackboard, Desire2Learn and eCollege. Some of these products are commercial while others are considered as open source software. Sometimes the open source software is misunderstood as 'free' software. To comply with open source license, the code must not only be free, but others must be given the right to modify and redistribute it for free. Choosing the right e-learning platform is a responsible and challenging task. In 2008 introduction of e-learning platform at University 'Goce Delcev' was a pilot project and decision was made to be implemented as open source software. Reasons behind this decision were several: low financial costs, system flexibility and expandability, low financial risk in case of failure of the whole project, possibility to use the already employed personal in IT department for software maintaining and administration. In higher education, Moodle's reputation also stems from the academic community's values of freedom, peer review, and knowledge sharing. Supporters say that Moodle helps educators create an effective collaborative online-learning community using sound pedagogical principles for a very low cost [1]. You can easily and quickly install it, it can scale up to accommodate a large user base, and it provides typical LMS features present in most similar commercial products. Moodle updates are common, the development community is very supportive, and its universal use is providing reliable learning solutions. All above mentioned advantages of Moodle have largely contributed towards its implementation as an e-learning platform at University 'Goce Delcev' and consequently at Electrotechnical Faculty.

2. Advantages and disadvantages of IT technology supported learning

E-learning platforms have transformed the ways the teachers teach and students learn. E-learning can be used as supportive instrument with blended learning or as a minimum instrument in distance learning. But in both cases learning resources must be interactive as need for interaction is one of the basic pedagogical principles and the domination of non-interactive resources is not helping in achieving good learning outcomes [2]. This transition of teaching methods had made it possible students to take part in the learning process, while the role of the teacher is that of "conductor" orchestrating and guiding students in education. Within this project, university professors have to modify the subjects and methodology involved in teaching/learning. In the process of realization of electrical engineering courses, we approach to the e-learning method developed in the following way:

- We created electronic courses, consisted of attached lectures and exercises as basic learning materials, supplementary materials, scripts, a collection of exercises and electronic books. The courses enable papers and homework to be attached.

- Computer exercises where simulation software is used for simulation of operation of electronic circuits are published as electronic workbooks attached in the electronic courses where each exercises can be prepared in advance by the student from theoretical point of view and results can be written and recorded in the workbook during simulation classes. One example of simulation exercise of diode rectifier and obtained results is presented on Fig.1.

- Speed of communication between professors, collaborates and students increased through the use of tools for collaboration and communication, setting up discussions forums etc…

In the process of transformation of lecturing we observed the following benefits:

- Lecturing materials as well as electronic books were available to students at any time free of charge.

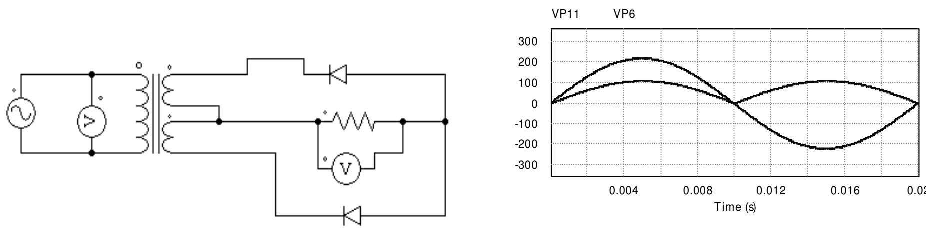
- The students are not forced to 'take' notes at lectures and exercises and they can become active participants in teaching process.

- In learning process communication with other participants is possible without physically meeting them which saves time and money.

In this process we met some difficulties in following nature:

- Using e-learning requires basic knowledge and computer skills. Without basic computer literacy, e-learning would be difficult. It is necessary to posses adequate computer equipment because the slightest technical problem will affect the student's concentration [3].

- E-learning requires student's greater responsibility. Students should by themselves estimate how much time they need for learning certain contents and they should motive themselves.



a) Circuit diagram b) Result from exercise

Fig.1 Example of rectifier circuit exercise implemented in Moodle platoform

7. 3. RESEARCH METHODOLOGY AND ANALYSIS OF RESEARCH RESULTS

First part of the study adopted a survey research approach. The research utilized questionnaire conducted among fifty one students at Electrotechnical faculty. Thirty one students were students of the final fourth year of study, fifteen were students of third year and five were students of second year. Purpose of the survey was to gather students response regarding several important issues related to implementation of web based e-learning: e-learning usage by the students, improvement of quality of learning process, faster and more reliable accesses to all relevant course information due to e-learning courses and influence of e-learning to achieved study results. Survey has given an insight in student opinion and their acceptance of e-learning as a supporting tool of traditional classroom learning. Second part of the study is based on the analysis of obtained achievements of students in February exam term for year 2012 regarding subject 'Circuits analysis' when the lecturing was realized with classical teaching methods compared to achievements of students in February exam term for year 2013 for the same subject when lecturing was supported by e-learning. Data processing is done by statistic package SPSS 17.

In Table 1 are presented results of conducted survey regarding use of e-learning by the students. Presented results are clearly pointing out that all participants in the survey 51 are using Moodle (100 %). There is no record that Moodle is not used by the students. Evidently, Moodle is very popular among the students and they have adopted it very easily. Further investigation in survey was focused on impact of e-courses and e-learning on quality of studying, achieved results and improved means of gathering all relevant information. In Table 2 are presented results of the survey regarding improved access to all relevant information from student's point of view.

Table 1. Moodle use

		Frequency	Percent	Valid Percent	Cum. percent
Valid	YES	51	100	100	100
	NO	0	0	0	0
	Never heard about e-learning	0	0	0	0
	TOTAL	51	100	100	100

Table 2. Faster access to all relevant information due to e-learning

		Frequency	Percent	Valid Percent	Cumulative percent
Valid	YES	49	96,078	96,078	96,078
	NO	1	1,96	1,96	98,038
	No opinion	1	1,96	1,96	100
	TOTAL	51	100	100	

According to the further survey results 78% of students confirmed that e-courses have positive influence to quality of learning and achieved results. Second part of the research is devoted to analysis of the data obtained from exams in February exam term at two different years 2012 and 2013 for subject 'Circuit analysis' when there was not e-learning course as a supporting tool to classical learning method and when there was a e-learning course respectively. Results from the exam in year 2012 are presented in Tables 4 and 5 and for the year 2013 in Tables 6 and 7 respectively. In Table 4 are presented general data regarding number of passed and failed students while in Table 5 are presented results regarding achieved grades from the exam. In Macedonian high education system student grades are on the level from 5 to 10 and for exam to be passed a minimum grade of 6 is needed. From Tables 4 and 5 it is clear that 92% of the students has passed the exam and the average grade from all passed students is 6,64 for year 2012. In Tables 6 and 7 similar analysis is repeated for year 2013 where percentage of passed students is 84,6 and the average grade is 7. Increase of average grade from year 2012 to year 2013 according to the results from Tables 8 and 10 is nearly 0.36 %.

Table 4. The achievements of students in February exam term for year 2012

	Frequency	Percentage	Valid percentage	Cumulative percentage
Passed	23	92	92	92
Failed	2	8	8	100
Total	25	100	100	

Table 5. Results of passed students in February exam term for year 2012

		Frequency	Percent	Valid Percent	Cum. Percent
Valid	5	2	8.0	8.0	8.0
	6	9	36.0	36.0	44.0
	7	11	44.0	44.0	88.0
	8	2	8.0	8.0	96.0
	9	1	4.0	4.0	100.0
	Total	25	100.0	100.0	

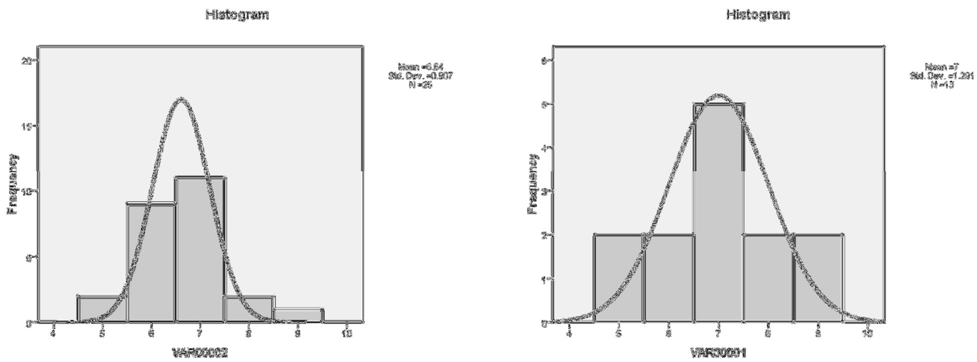
Table 6. The achievements of students in February exam term for year 2013

	Frequency	Percentage	Valid percent	Cumulative percentage
Passed	11	84,6	84,6	84,6
Failed	2	15,38	15,38	100
Total	13	100	100	

Table 7. Results of passed students in February exam term for year 2013

		Frequency	Percent	Valid Percent	Cum. Percent
Valid	5	2	15.4	15.4	15.4
	6	2	15.4	15.4	30.8
	7	5	38.5	38.5	69.2
	8	2	15.4	15.4	84.6
	9	2	15.4	15.4	100.0
	Total	13	100.0	100.0	

On Figure 2 is presented distribution of function which presents the achieved grades from February exam term in years 2012 and 2013.



a) year 2012 b) year 2013

Fig. 2 Distribution of function from achieved grades in years 2012 and 2013

In order to see weather the results from exams are improved when e-learning is used following hypothesis are set:

- Hypothesis H_0-there is no significant statistical difference between student achievements when e-learning is used and when is not used.

- Hypothesis H_1- there is significant statistical difference between student achievements when e-learning is used and when is not used.

Hypotheses are tested with Chi-Square test (χ^2). Results from testing are presented in Table 8. In order zero hypothesis H_0 to be accepted it is necessary parameter Asymp. Sig. to be bellow 0.05. In contrary, zero hypothesis is rejected. According to the performed test of χ^2 based on the data from Tables 5 and 7 parameter Asymp. Sig. is 0,028 which mean that zero hypothesis is rejected and alternative hypothesis H_1 is accepted. The Chi-Square test has proved that there is a significant statistical difference between learning outcome when students are using e-learning and when they are not using it.

Table 8. Test Chi-Square

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-Square	10.888[a]	4	0.028
Likelihood Ratio	14.551	4	0.006
Linear-by-Linear Association	.000	1	1.000
N of Valid Cases	13		

4. Conclusion

Educational system in Republic of Macedonia has gone under major changes during last decade. Competition among state and private universities and huge expansion of IT services and web technologies have led to new and innovative services offered to the students. One of those services is e-learning implemented at University 'Goce Delcev' on Moodle platform since 2008. Results form conducted survey among students has proved that all of the students (100%) are using Moodle in any form. Results from statistical analysis have proved increase of average exam grade of 0.36 percent when e-learning is used compared to the case when it is not used. These facts give a further impulse to expand even more the IT services offered to the students since they are accepting and evaluating them highly positively.

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Application of Matlab/Simulink in hybrid stepper motor modeling

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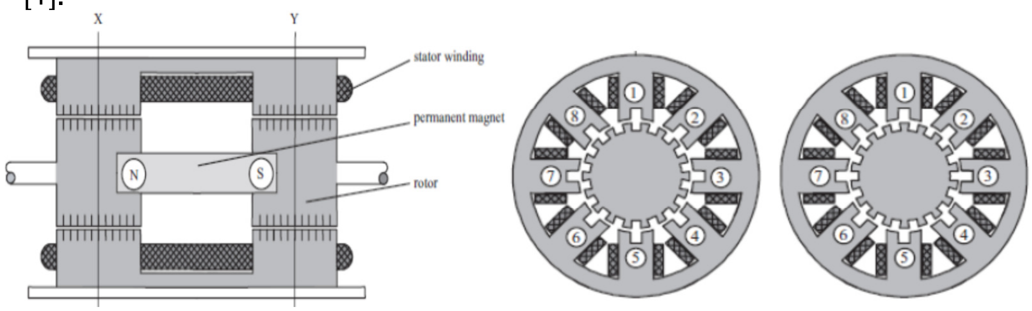
Abstract: *Development of digital electronics and microprocessor systems has led to development of electrical motors capable to be digitally controlled. These motors are widely known as stepper motors and the enable transformation of pulsed electrical excitation into mechanical energy. Matlab/Simulink is used as a simulation tool for hybrid stepper motor enabling motor transient characteristics of current, voltage, torque and speed to be obtained. Different operating motor regimes are simulated as no-load and rated load operation. Adequate conclusions regarding motor performance characteristics are derived.*
Keywords: *hybrid stepper motor, Simulink, motor's operating regimes*

1. Introduction

Stepper motors are very important in robotics, process control and instrumentation. They enable precise control of motor angular position, speed and direction of motor rotation. They are capable of discrete precise movements i.e. movements in precise steps so they are named as 'stepper motors'. Stepper motors are transforming electrical energy (excitation) into mechanical movement. They are constructed as rotating or translating motors. Although stepper motor are known for a long time, they have achieved their wide popularity in the last thirty years due to development of electronics which enables construction of cheap and reliable control circuits capable to satisfy complex requirements regarding motor torque, speed and angular displacement. In order their transient performance characteristic to be analyzed Matlab/Simulink is chosen as simulation tool and motor characteristics are analyzed under different operating regimes: no-load, rated load and over load. Advantages of stepper motors are: low costs, small dimensions, possibility to transform the pulses from digital inputs into angular movement-step, number of steps is equal to the number of control pulses. The above mentioned advantages have lead to their wide application in control systems and robotics and have made them irreplaceable moving force of industrial processes [3].

2. Hybrid stepper motor-construction and principle of operation

Hybrid stepper motors have magnetic core which is excited by combination of electrical windings and permanent magnet. Electrical windings are placed on stator while rotor is made of permanent magnets (Fig. 1). Number of poles at stator are usually eight and each pole has two to sex teethes. Per pair of poles are placed two excitation windings for example one winding for pole 1,3,5 and 7 and another for pole 2,4,6 and 8 [1].



a) side view b) cross-section view

Fig. 1 Construction of hybrid stepper motor

Length of each step can be calculated if number of rotor teethes are known-p. For hybrid stepper motor length of one seep is calculated from:

$$\text{Steep length}=\left(\frac{90}{p}\right)^{\circ} \qquad (1)$$

Hybrid steeper motor has small steep, typically 1.8° and larger torque compared to variable-reluctance stepper motor. These two parameters can have a decisive advantage when there is an application with limited operating space.

3. Simulink Model Of Hybrid Stepper Motor

Hybrid stepper motor is operating due to electronically commutated magnetic field which enables rotor movement. All excitation windings are placed at stator while motor rotor is constructed of permanent magnet or soft magnetic material. In Fig. 2 is presented block diagram of motor simulation model constructed of three basic blocks: controller, driver and motor.

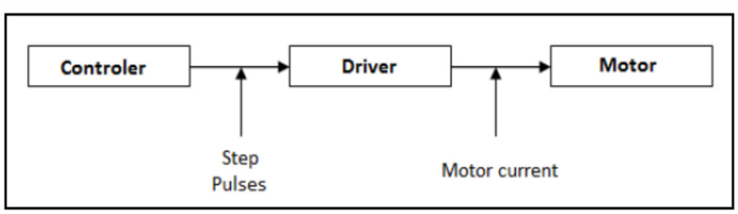


Fig. 2 block diagram of stepper motor

Simulink model from Simulink demo library is presented in Fig.3 and it is consisted of two sections: electrical and mechanical [4]. According to Simulik model motor input parameters are: phase voltage (A_+, A_-, B_+ and B_-) and mechanical load $-T_L$.

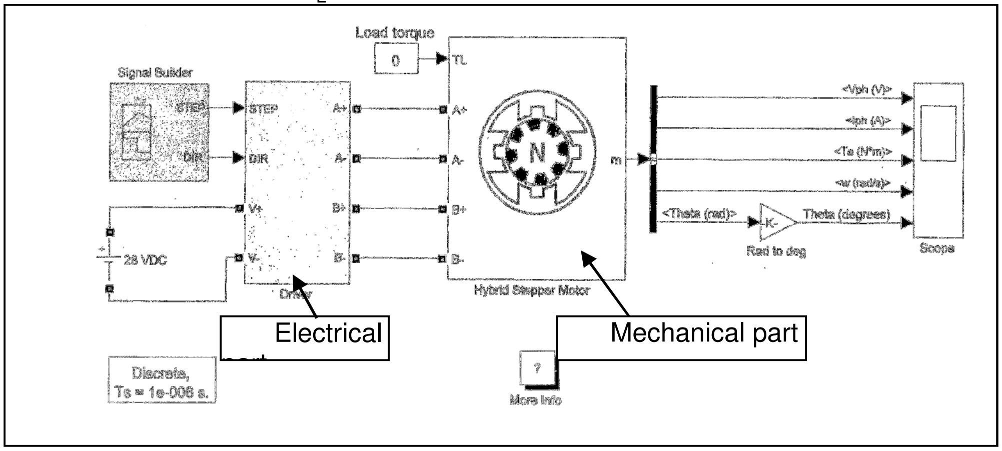


Fig.3 Simulink model of hybrid stepper motor

Output parameters from motor model are: phase current-I_{ph}, electromagnetic torque-T_e, rotor speed-w and rotor position-theta. Electrical part or motor control circuit is consisted of three functions entities: control block, hystersis comparator and MOSFET PWM converter (Fig. 4).

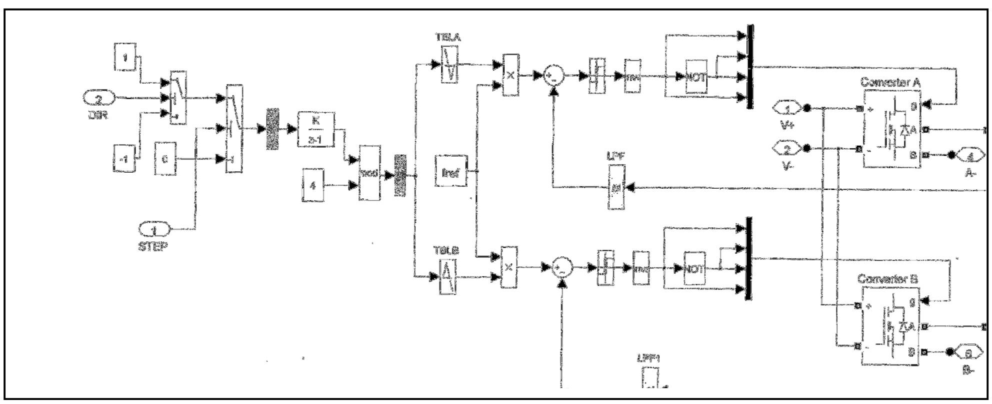


Fig. 4 Simulink model of control circuit

Motor movement is controlled by two signals: STEP and DIR which are output signals from block Signal Builder. Positive value (value of ‘1’) of signal STEP enables motor rotation while value ‘0’ stops the rotation. DIR signal controls the direction of motor rotation. Positive value (value of ‘1’) enables rotation in one direction while value of ‘0’ reverses the direction of rotation. Converter bridges “A’ and “B’ are H bridges consisted of four

MOSFET transistors. Bridges are supplied by 28 V DC and their outputs supply the motor windings with excitation current and enable the motor movement.

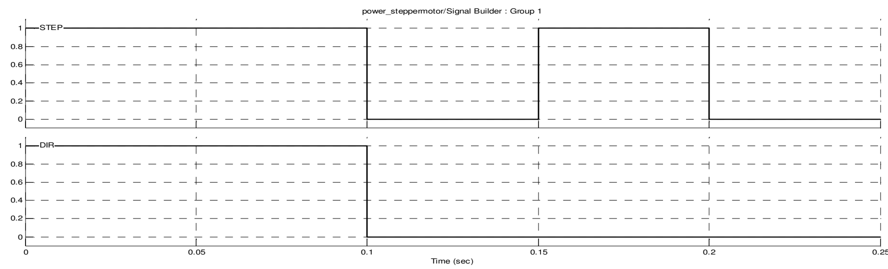


Fig. 5 Output signals from Signal builder block

4. Simulation results

After all motor parameters are input in motor model simulation is run. Time for simulation execution in is defined to be 0,25 seconds according to the signals from Signal Builder block and set time in Simulink model. First simulation is run at no-load operation or motor is running without any load. From the simulation results presented in Fig. 6 it can be concluded that motor is moving in one direction for 0,1 seconds (STEP=1 and DIR=1), stops in period from 0,1 to 1,5 s (STEP=0, DIR=0) 0,05 seconds is rotating in opposite direction (STEP=1, DIR=0) and again it stops for 0,05 seconds (STEP=0 and DIR=0). Motor transient performance characteristics are presented in Fig. 6 for no load operation.

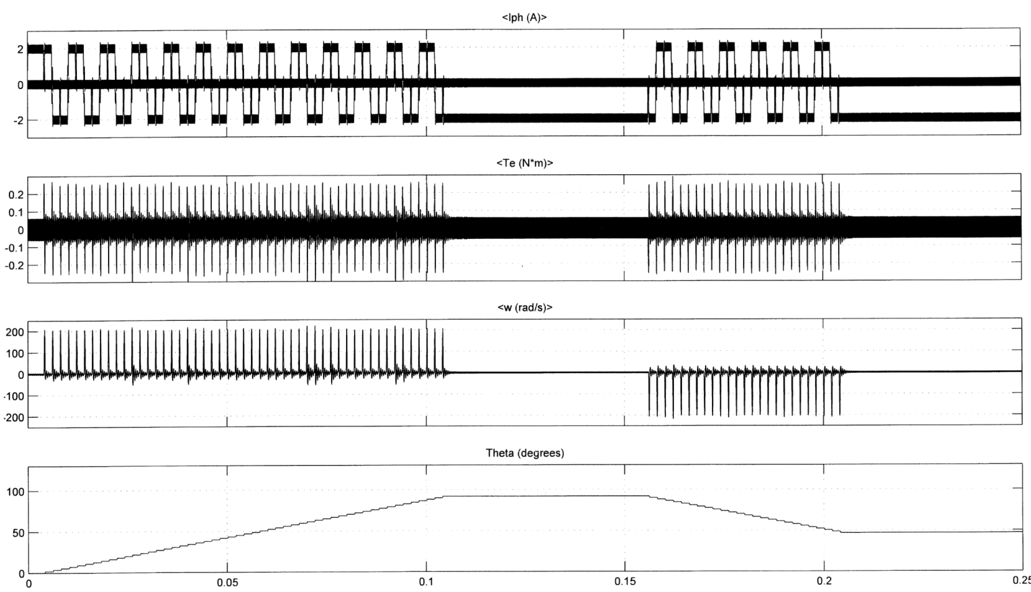


Fig. 6 Motor transient performance characteristics at no-load

With adequate zooming of presented results in Fig. 6 it can be noticed that motor has reached the speed of 200 [rad/s] and have moved from position 0° to 98 degrees. It remains in that position for 0.052 seconds before it starts for time of 0,156 to move in opposite direction and it stops for time of 0.204 seconds on position 47°. For case that load torque is increased to value of 0.1 Nm motor transient characteristics are presented in Fig. 7. In the same time motor moving sequence is changed by changing the values of step signal to 1 for 0-0.15 s, and from 0.2 to 0.25 s. In the interval between 0.15 and 0.2 its value is zero.

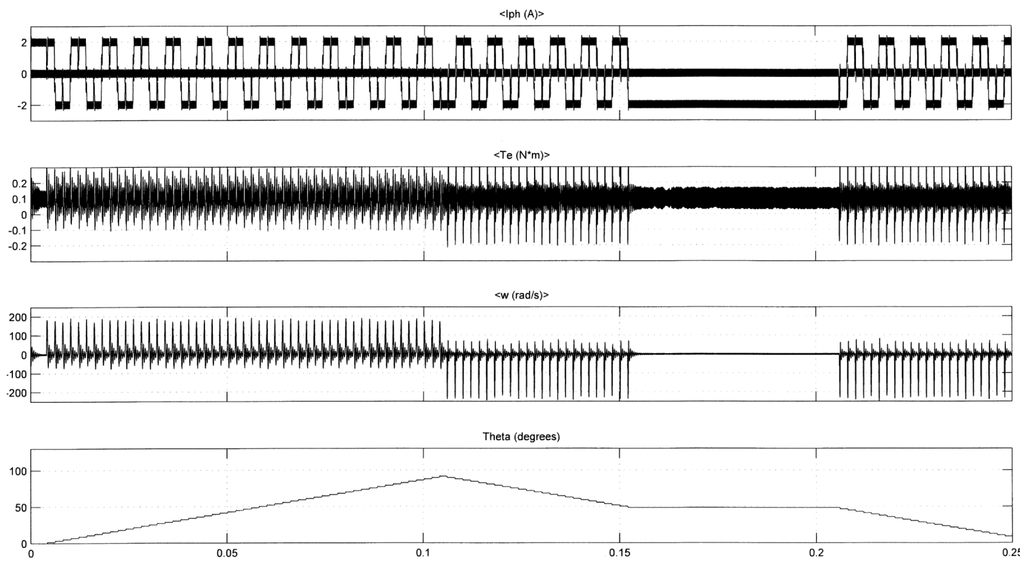


Fig. 7 Motor transient performance characteristics for load 0.1 Nm

From presented results in Fig.7 as it is expected there is a change in motor transient characteristic of electromagnetic torque. Its value is increased near to 0.1 Nm in order to be able to drive the load torque. For case that load is further increased to 0.4 Nm simulation results are presented in Fig. 8.

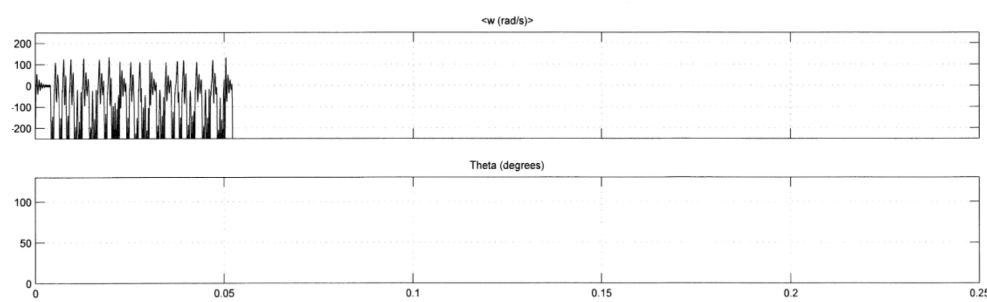


Fig. 8 Motor transient performance characteristics for overload of 0.4 Nm

Since applied external load is bigger than then motor electromagnetic torque motor is not capable to move the shaft coupled with the load so motor speed-w very shortly after the motor start is going to zero (0.05 seconds) and motor angle of movement is zero. In case that number of motor steps are changed from 500 steps/second into 150 steps/second at no-load operation, simulation results are presented in Fig. 9. From Fig.9 it can be concluded that steep length is bigger due to decreased number of steps, final step position is 25° in one direction and 13° in opposite direction.

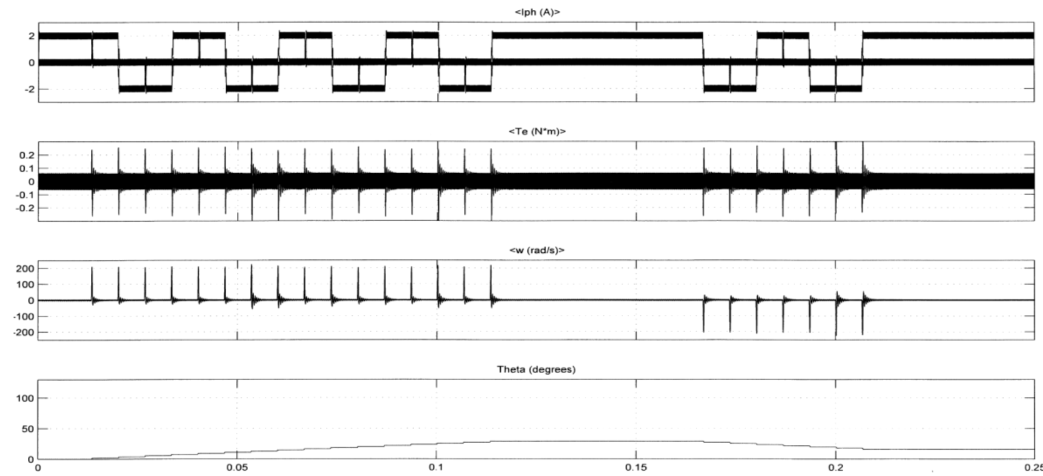


Fig. 9. Motor transient performance characteristics for 150 steps /second

5. Conclusion

Different simulation software packages during recent years have proved itself as a useful tool in analyses of electro engineering problems. Simulink with its extensive block libraries enables wide possibilities for electrical machines simulation. In this paper is analyzed simulation of hybrid stepper motor transient performance characteristics under different operating regimes: no-load, rated load and overload. Simulation results proved that motor is running in forward and backward direction according to the applied signals from PWM inverters to the excitation windings and only in case when applied load is smaller than motor electromagnetic torque. In case when external load is bigger than motor electromagnetic torque no rotor movement is achieved and motor speed is rapidly going to zero very shortly after motor start. Application of simulation packages has considerably improved electrical machines analysis replacing the expensive laboratory equipment and enabling performing of different experiments easy and with no cost.

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Online generation of psychological tests

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Abstract: *In this paper we present a system for online research in the field of psychology. The created system could be successfully used to generate other types of tests – didactic tests, achievement tests, IQ tests etc. Online testing provides a large field for scientific research. The following work which is part of the master thesis in the subject of informatics from the South-West University in Blagoevgrad, Bulgaria deals the possibilities of online testing.*

Keywords: *computer administering, online system, psychological testing, computerized psychological assessment, personality questionnaire,*

1. INTRODUCTION

Computers have become a valuable tool in the last couple of years in all the stages of psychological testing – from the construction of the test to its implementation, the calculation of the point average, calculation of the results and their interpretation. [2, 3]. The evident increase in the speed of analysis of the data and the calculation of the variables are some of the advantages in using computers. Of equal importance is not only the facilitation of the testing process but also the opportunity to develop new methods and approaches in the field of psychological testing which could not exist without the speed and calculative powers of modern computers. The need of symbiosis between informatics and computer science and humanities and social sciences is thoroughly presented in [6, 11].The newest tendency in psych diagnostic research connected with the use of electro-calculative technology – computererized testing was the result of such interdisciplinary interaction. Testing through Internet is a type of computerized testing and although it appeared just recently it is becoming more and more popular.

In the following paper we present an online system for generation of computerized tests. This system is meant for creating of personality questionnaires, but it can be used to the same extend for the creation of other types of tests such as didactic tests, achievement tests, IQ tests etc. What is convenient in this case is the possibility to create a large database with various test values (items) and their connections with the different psychological, didactic, scientific etc. categories and definitions. The process of selection of the various items from the database during the creation of a given test is also automatized. The biggest advantage of the online system is that it can be used by whole collectives and target groups. In order to secure the system against malevolent meddling and misuse an electronic defense is set up, where the different types of users have a different types of security clearance.

, The theoretical part of the development is borrowed from these publications [5, 7, 8, 10].

2. PRELIMINARY

Psychological assessment will be called any psychological testing made with the help of a preliminary prepared test – a list of questions or statements, that the assessed person or group of people has to answer or to give their opinion for. The separate parts of the test are called *items*.

We will call *computerized psychological assessment* any psychological testing, where one or several phases of the testing are made with the use of a computer.

Computer administering of psychological tests is the computer representation of the entire procedure of psychological assessments – test construction, test implementation, results evaluation, storage and maintenance of the developed database, its statistical processing, analysis and interpretation.

Personality questionnaires are those psychological tests, which are purposed for description and evaluation of the characteristics of cognitive (behavioral), emotional and motivation sphere, the interpersonal relations and attitudes of an individual [12, 13, 14]. It's typical for the Personality questionnaires (in contrast to Achievement tests or Intelligence tests) that the items are questions or statements, for which answers the respondent has to report certain information concerning himself, his experience and relations. The format of the answers to items is also specific – most often they are described with the help of a finite set of preliminary known answers or statements, which we will mark by Ans. In our examinations we will accentuate mainly on finite sets of possible responds to each item of the test. For example $Ans =$ {"*Yes", "No"*}, $Ans =$ {"*True*", "*False*"}, $Ans =$ {"*I like it*", "*I don't like it*"}, $Ans =$ {"*Often*", "*Sometimes*", "*Never*"}, $Ans =$ {"*True*", "*I don't know*", "*False*"}, $Ans =$ {"*Agree*", "*I'm not sure*", "*Disagree*"}. Items with rating scales are also used in practice, but we will examine just rating scales that represent definite and consequently discrete set of real numbers.

For more details about basic terms in personality psychological testing see for example in [1, 4].

3. ADMINISTRATOR MANUAL

3.1. Control panel

Access to the control panel is granted only to the system administrators. It provides opportunity for generating of psychological tests, definition of psychological categories, and assignment of possible answers.

a. “**Start” menu** – contains general information for the purpose of the system.

b. “**Create test” menu** – this menu provides the opportunity to generate psychological tests. The following information need to be put in:

- Name and description of the test – the name and the description of the test are the first things which will be visualized. When a common user logs in the system and requires to make a research, in a certain category, they will have the right to choose among the tests already put in by the administrators which correspond with the desired category.

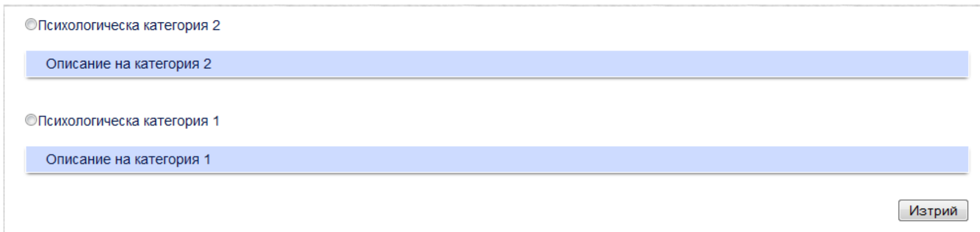
- Choose category – each test belongs to one or more psychological categories. The test you input will belong to one or more psychological categories which you must choose and then mark with a tick. If the required category is not listed, it can be put in via the „Category“ menu.

- Please choose the answers which you want to be part of the test – with a radio button you can choose the answers which correspond with the psychological test that is put in. If the required answers are not listed, they can be put in via the „Answer“ menu.

- Please put in the number of items – In case the psychologocal test that is put in contains **n** number of items they need to be put in successively. In order to manually do this, put in **n** in the “Please put in the number of items” field. After pressing the OK button 30 fields are automatically generated in which you can put the items you require. After pressing the “Save” button the test will be generated.

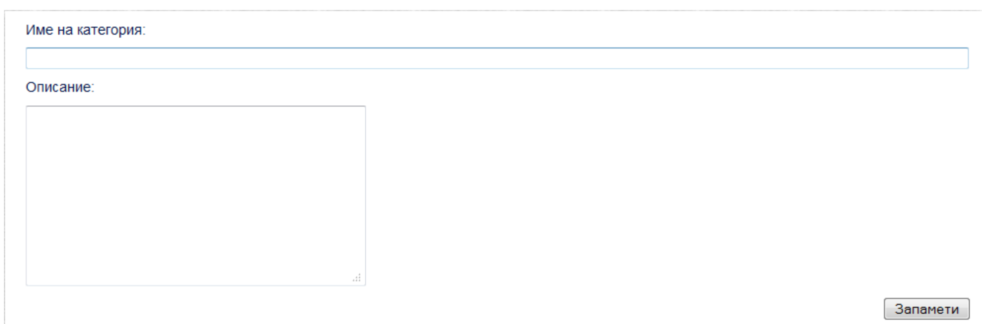
c. **“Delete test” menu** – the already existing tests can be viewed in the “Delete test” menu in case they need to be deleted.

d. **“Category” menu** – each test belongs to one or more psychological categories. The already existing tests and their descriptions can be seen in the “Category” menu:



Each psychological category could be deleted. To do this just mark the category and press the “Delete” button.

In case you want to create a new psychological category put in the required information and press the “Save” button.



e. **“Answers” menu** - in each psychological test there are answers which are the same for all the questions. If for a certain test “Yes” and “No” are required, go to the “Answers” menu and put 2 in the “Please put in the number of answers”. After pressing the “OK” button 2 fields will be automatically generated, where the required answers can be put in. (in this case “Yes” in the first field and “No” in the second field).After pressing the “Save” button the answers that have been put in can be used for an unlimited amount of tests. The answers that were put in are visualized and separated in categories depending on their number:

If you wish to delete the answers, mark the radio button and press the “Delete” button.

4. CONCLUSION

The present paper is part of the master thesis of the students Gergana Praskova and Ivan Petrov from the Faculty of Informatics under the mentoring of PhD K. Yordzhev and the professional consoling of PhD of Psychology Ivelina Peneva. This paper presents interdisciplinary characters and can be seen as proof for the wide area of expertise and the possibilities for professional realization of the students, who graduate this faculty. The aforementioned could also realize their professional potential in the humanitarian fields such as the psychological research and their computerized processing.

The system developed from the students is meant for psychological research, but it can also be developed and implemented for the use of pedagogical purposes and for the evaluation of achievements in any human field of activity.

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Entertaining problems in the teaching computer programming

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Abstract*: The report presents selected entertaining problems that can be used in the programming courses. The problems are grouped according to the mental activity that encourages - algorithmic thinking, structural thinking, heuristic thinking.*
Keywords: *algorithm, problem, computer programming, teaching*

1. INTRODUCTION

The computer programming is a process of creating software using a programming language. The purpose of the training of computer programming is to acquire basic knowledge and skills that can be applied for designing efficient algorithms for solving problems and to select the most appropriate data structures.

2. PURPOSES AND TASKS OF THE TRAINING OF COMPUTER PROGRAMMING

Unlike the real programming the educational programming passes through several branching actions, so that the transition between them depends on the results of the task:

- an initial formulation of the problem
- an algorithmic description of the problem
- programming problems of high-level language

For realization of the educational intents need to be selected appropriate programming problems with which to develops the thinking of the students. The ability to make an algorithm for solution of the problem is called an algorithmic thinking. The ability to split the problem into blocks that have only one point of entry and exit is called structured thinking. The ability to find the truth as to prove the proper solution of the problem is called heuristic thinking.

To maintain students' interest and faster comprehension the ideas of the methods, is necessary to diversify the teaching with interesting problems. The entertaining problems attract attention, awaken a curiosity using unusual in their content, entertaining, paradoxical phenomena, events or results. In most cases, although they have original themes, they are based on the same algorithmic solutions

3. PROBLEMS THAT DEVELOPS ALGORITHMIC THINKING

To solve problems from different origin is needed to establish an action plan to achieve the desired result. Entertaining problems enliven the lessons, they increasing the interest in the theoretical aspects of the programming. Problem 1.1. is from integer arithmetic and can to solve with the instruction cycle.

Problem 1.1.

Two players play the following game. The first player chooses a number from 1 to 9. The second adds a new digit and reports the sum. The game continues until the sum becomes equal to the number X, which they chosen previously. The player that received the sum X is the winner. Choose a winning strategy! Who player should to start the game so as to be winner?

Algorithm:

```
int I=0;
do
{X=X-10;I++;}
while(X<10);
```

If I is odd, the first player wins, otherwise the second is winning.

Such reasoning we apply for the solution of the Problem 1.2.

On the blackboard is written K symbols "-". Two players take turns and everybody have the right to substitute one “-" with one "+" or two consecutive "- -" with "+ +". The player for who did not remain "-”, loses the game. What is the winning strategy?

The classical structure array is used to realize the idea and the decision of the next problem.

Problem 1.3.

Each of the residents of a city is a member of one tour group. Once a year, each group chooses a country to visit. If several groups choose one country, they unite. If they do not specify a new desire remains in force the choice from the previous year. Determine the composition of each tour group!

Algorithm:

We Introduce N(number of inhabitants) and M(number of countries). The countries are numbered and saved in an array A []. The residents are also numbered and in array B [] for each resident, we are saved the number of the desired country.

```
for(int I=0;I<M;I++) A[I]=I;
cin>>K; // number of a city from the previous selection
```

```
cin>>T; // number of a city from a new choise
for(int I=0;I<N;I++) cout<<A[B[I]];
```

In Problem 1.4. with the type string expands the set of used types.

In the Bulgarian alphabet some letters match the letters of the Latin alphabet. From input text written with Latin letters we must find the words that we can consider that are written with Bulgarian letters.

Algorithm:

For each word we check are there a letter that is not Latin.

4. PROBLEMS THAT DEVELOPS A STRUCTURED THINKING

The structural style of thinking is based on the ability to split the task into subtasks, build a tree of nested subtasks to identify local and global objects for each subtask, clarify the question of the exchanging data between subprograms. The interesting and original problems diversify the educational process and help fast perception the ideas of the method. The principles of structural style of thinking can be applied in the solutions on the following problems.

Problem 2.1.

How many ways a bank employee can submit one amount in banknotes of 1 BGN, 2 BGN, 5BGN., 10 BGN, 20 BGN, 50 BGN, 100 BGN.[1]

Algorithm:

We divide the iterative solution to the following subtasks: Problem 2.1.1. - count the possible representations of the original sum; Problem 2.1.2. - generate an array that saves the number of banknotes in each crushing; Problem 2.1.3. - from the number of the crushing we are receiving the banknotes that involved in the presentation of the amount;

Problem 2.2.

Two players select arbitrary date. Each of the participants on his move increases by 1 or 2 number of day or month, but not both. The player who reports December 31 loses the game.

Algorithm:

We divide the task into three subtasks. Problem 2.2.1. - filling the binary array bool A[30][11] with 1(winning position) or 0(losing position) for each day of the year; Problem 2.2.2. - check whether the player comply with the rules; Problems 2.3.2. - report the winner

Problems 2.3.

In a scientific experiment N (odd) objects participate. Each of which has a weight W (integer). Find the object whose weight (W_{ob}) meets the following requirements: The numbers of objects with a weight less than W_{ob} is equal to numbers of objects with a weight more than W_{ob}.

The weights can compare only with a device capable from three different objects to determine who of them meet the condition.

Algorithm:

We divide the task into three subtasks. Problems 2.3.1. - introduces a number of objects; Problems 2.3.2. - from three different objects that come as input parameters, select one that meets the condition; Problem 2.3.3. - shows the number of search object.

5. PROBLEMS THAT DEVELOPS HEURISTIC THINKING

The Heuristic style of thinking is highly developed intuitive thinking that facilitate the generation of new ideas for solving unconventional problems such is not always the solution is optimal. The rules for solving originative problems are called heuristic rules. The basic idea for solving heuristic problems is intuitively to choose one of all possible cases based on an optimality criterion. The heuristic methods are varied and can't describe by a general scheme. Often they apply with the method of the complete search to reduce the number of variants examined.

Problem 3.1.

The weights of white (W) and black (B) balls perform several dependencies. Three white balls are heavier than four black balls. Two white balls are lighter than three black balls. The weights of W and B are odd numbers and less than 50 g. The sum of the weights of three white balls and one black ball is more than 170 g. Find the number of solutions of the problem.

Algorithm:

```
N=0;
for(int I=0;I<25;I++)
for(int J=0;J<25;J++)
{X=2*I-1; Y=2*J-1;
A[N]=X;B[N]=Y;
If(3*X+Y>170 && 3*X>4*Y && 2*Y<3*X) N++;}
```

6. CONCLUSIONS

The described entertaining tasks are a powerful tool for learning the skills for reasoning and development of logical and abstract thinking. They can be used in the computer programming courses for students, for pupils from the secondary schools and for preparation for competitions.

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THE SYMMETRY – A FUNDAMENTAL PRINCIPLE OF THE PYTHAGOREAN MODEL OF THE COSMOS

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Annotation: *In the study is presented a systematical and original interpretation of Pythagorean views. The are grounded heuristic aspects of the link between the principle of symmetry and Pythagor's ideas about numbers, measure and harmony by resuming Pythagorean views about the ten like a perfect number. The gnoseological transcend from a number– measure– harmony – symmetry is realized by an original way.*

Key words: *number, measure, harmony, symmetry, cosmos, decade.*

The tendency to associate the rational believes of the symmetry by the content of the ideas of the harmony of numbers is clearly stands in the earliest written sources of the ancient culture. Even in the works of Homer the use of the term "number" "has esthetic and even special meaning for the formulation of the principles of symmetry"*1. Here the number and harmony are still introduced as the same semantical meaning, as the same verbal distinctness.*2, but in the early Pitagoreans a definite tendency for their total differentiation as mental abstracts showing the different sides of reality's phenomena. They/the early pitagoreans –Я.Б./ as Alexander says, have been assuming to be obvious and have been proving that the sky/universe/ is compounded by numbers according to the harmony.*3 In the naturophylosophical charters of the early pitagoreans, symmetry is no longer no longer associated with the properties of single numbers but as a characteristic of the specific numeric relations: symmetry is equal to commensurability, to commensurability and proportionality. Pitagor was laying as basis the numbers and the number proportions **(simmetria – συμμετρία),** which he also calls "harmonyes"*4.This fact is clearly determined by the strictly proportional and spatially-configurational believes for numbers in which the geometrical and not to mention the arithmetic was never separated from the physical"*5.

The relations between numbers are presenting the naturephylosophical study of the pitagoreans as spatial forms, therefore “material”, strutures to be perceived by the senses, organized in needed and therefore in mandatorial and just order. For them the world is not just a Universe ,not just ”Sky”, Cosmos, but a certain type of chaos frozen in harmony, invariable order of numerical relations, natural symmetry of the complete perfect whole.

A.N. Pavlenko writes “The Cosmos is on one side mathematically found and proven mathematical proportion – it could be said in military manner – “Code”, organizing the world according to this “Code”, because at first the term “cosmos” ment strictly lined military formation.On the other hand cosmos is the beauty of the world, it’s magnificence.”*6. In this model of the cosmos order ad symmetry are equal and are completely the same: Order and symmetry are wonderful and useful , disorder and asymmetry are useless and harmful”*7

In the naturephylosophical treatis of the early pitagoreans, the cosmos is displayed in a stable and steady state. In it there is movement but no development. The cosmos is invariable in its quantitative and qualitative distinctness for them and the concept of symmetry is identical of the concepts of balance and durability and if accepted as definition of the signs and originality of perfection and the unity of matters. This understanding of symmetry influences most significantly the building of the pitagorean comogonia and its main idea of the non moving “central fire” and the earth moving around it and around its axis.”*8

Therefore the concept for Symmetry moves in the process of the development of ancient culture transforms from being intuitively clear and understandable to being output in the creation of many philosophical models of the world , having gained new rational and gnoselogic importance. Symmetry turns into the basic principle of intellectual constructuration of theoretical work thesocial importance of it’s results is guaranteed from the belief of the correctness of the methods of abstract shemes of phenomenons, and the belief of the truthfulness of human thought is drawn not from phenomenons/observations/ but more often from thought, as the pitagorean discographies say”*9

In the ancient greek thinking the theoretical development of the problem for the “truthful” knowledge is connected the problem for the “wonderful”, the solution of which would be unthinkable without the knowledge of symmetry and harmony. This specificity of the interrelationship is distinctively expressed in the recreated by Yavlih structure of the pitagorean questions, in which central place take the following: Which is the wisest? – The Number. Which is the most wonderfull? – Harmony.*10 ” A.F. Losev writes that in the very concept of harmony, it’s mandatory to have contained, and still containing a moment of positivity , moment of approval , moment of judgment, moment of truth moment of independent reality… if not, harmony would not be a subject of aesthetics, would not be something wonderfull"*11

The work os R.Brambo is cleared and more directly formulated:

"The unifying beginning of the scientific interests f the pitagorean school and those chapters of their religion alike, preaching a definite way of life was a constant use of harmony and order as a criteria for truthfulness and beauty in nature as well as in human relations."*12 For the pitagoreans the decade is wonderful because it is great and perfect'*13,"from the theoretical and practical sciences, wonderful are only those which are touched by love for the wonderful"*14, m truthful knowledge is possible only in " a wonderful look at matters as they are in their properties"*15, and soon. The wonderful look in the studies of the Pythagoreans is identical to the proper sensory perception and it's accepted as a mandatory condition for the accomplishment of the truth, because the truthful and the wonderful are usually the same in the reiview of usefull objects. A.N Pavlenko realllya accurately fixates this feature in Pythagorean world awareness "Esthetics with its sensible perception of reality with its variety of spiritual experiences, in some way turns out to be complimentary with purely rational perception of reality, which the in ancient Greece was associated with the concept "logos", the ancient Pythagoreans – with ancient mathematics.*16 "The theoretical review of matters by the Pythagoreans includes the contemplation of beauty as well as of truth and only after a definite period of is differentiated to be epistemological: and esthetic:*17 "The wonderful" in the Pythagorean is associated with their naturephilosopical postulate for the existence of objective beauty, which they call symmetry. Objective beauty is "doing of measure and number" and it's accomplished by the sensory contemplation, as a way of aesthetic experience, and by the mind in as kearning of the "number harmony" and the numbers'proportion. Yamvlih reports that in the Pythagoreans' opinion, one who wants to study the existing and its proportions must look at the numbers, at the variables of the existing and the proportions, because by them everything can be explained."*18 The creation of a rational conception for symmetry in the Pythagorean studies is possible only in the theoretical thouths for the number relations and the specific intellectual sense of numbers.

The historical contribution for the foundation of the mathematical understanding and expanationof the world, because they first come to the conclusion that"the book of nature is written with the language of mathematics"as Galileo would say two thousand years later"*19, the numbers and their relations are viewed as a key to the understanding and of its structure."*20 and their huge contribution in the treasury of the world's science is defined mostly with the meaning of their study of the number.*21 This study shouldn't be fully taken to this part of mathematics known today as" the theory of numbers". Pythagor's study of the number is more like a philosophy of the numbers and excludes everything that ever since the time of Plato is called logistics “*22. But in this philosophy a lot of mystical and speculative moments, it is mixed with rough and burdensome sperstición"*23 and it outlines one of the specifics of the whole ancient Greek philosophy. The combination of theology and mathematics is started by Pythagoras, P .Rusel generalizes that it’s common for the philosophy in the whole ancient Greece.”*24 This mixture of phytagoriang philosophy and mystic is a product of the absence of strict differentiation of the scientific searches of Pythagoras and his followers from the mystical cultural practice of their religious union. Some ancient discographers even thought that the mathematic excersizes practiced in the Pythagorean union were fully associated with the religious practice of a cult:”Prokl says: The mathematic science was invented by the Pythagoreans to remind /anamnesis/ of the godly; by then and by images they were trying to transcend to the beginnings of the beyond. As it was said by those who were dealing with exhibitions of Pythagorean studies, they dedicated to the gods, the numbers and the figures.”*25 This quote has important methodological meaning, because it gives the key to the disclosure of the complicated and dynamic hierarchy of the different composing the pythagoreansy as a complicated, but unitary religious, political, metaphysical, mathematic, scientific, esoteric, pedagogic and aesthetic phenomenon in the history of antiquity, key accurately fixating the value and the rising and falling the self relaying role of any of these activities, which were inherent to the union trough out its entire history*26 with their place in the achievement of the higher purpose of the Pythagorean union. This higher purpose is defined by the common metampsychic orientation of the Pythagorean world seeing and it’s reduced to the salvation of the soul by the complicated, two-staged, secret assimilation of the rules of the Pythagorean way of life , strictly regulating all sides of the life of any of the union’s members / the very amendment of the purpose of the pythagoreansy in the history of antiquity will be examined later on Y.B/. In this process of achieving the higher purpose of created by pythagor new religion the higher activity in the hierarchy of the forms of activity, belonged to the religious one.*27 ’The last one was clearly the most precious for those members of the union, writes A.O. Makovelski, who were at the first stage of mastering the doctrine as a whole, on almost philosophic degree ‘*28, and for the scientific exercises was left the honorary place / in relation with other forms of activity – gymnastics and music –Y.B/, but still an insignificant place ..,,*29 The same Russian philosopher, proceeding from the traditional division of the members of the Pythagorean union into two groups – higher- mathematicians and – lower – akusmatics*30 /H.Dills doesn’t view the akusmatics and the mathematics as two consecutive degrees of achieving Pythagorasy, he presents them as different types: a/the akusmatics were fans of Hipas, who was banned from the union , who thought that the basis of the world was fire and they were strict followers of the rules of the pitagorean way of life , regulated of the questions and the religious rituals, but they were paying less attention to the educational and theoretical activity; b/ the mathematicians – they were the real followers of Pythagoras's work, sacred to full depersonalization, guarding and improving the authority of their teacher, giving him the credit for their discoveries, giving priority to the scientific activity over the following of the akusmatic rituality, possessed by feeling of intellectual supremacy over the akusmatics and the right to lead in the political activity, with witch they caused wrath and angry contempt – Y.B,/,formulates the reigning in modern history of ancient philosophy thesis, thepythagoreans considered metaphysics as a higher form of activity, studying it would require access, given only to those members of the union who had learned the lower truths.*31, /self scientific activity, associated with the learning of private knowledge , not their further development.*32 . On on hand the mind of the Pythagorean is had not yet formed an abstract, mental, rational concept for the number and the number relations in an mysticaly-religious view of numbers. Ïndependant from a deffenite sharpness of the mathematical abstractions, writes D.V. Joehadze, the pythagoreansy stays basic in the parameters of beliefs and meditation."*33 R.Brambo's conclusion is similar: "Yet in the Pythagoreans, nevertheless the often celebrated grace of evidence and definitions, the numbers were a lot more associated with the imagination unlike today's abstract numbers."*34 The absence of an abstract concept for the number gives birth to apriorisum. "Theoretically numeral operations, announces A.F. Losev, are appreciated as a most necessary apriorisum, which is obvious and not needing to be proven."*25 On the other hand, "for the Pythagoreans the number is also an image of matters, symbol of the sensory perceived objects, and thought, and concept for their ratio, and the essence of their existence and a principle of their progress and change"*36

In the Pythagoreans studies , the numbers is represented as a fundemantel principle of organization of the cosmic, overcoming chaos, harmonic order…"And because for the beginning of this harmony were considered numbers, Alexander comments, then ofcourse for the beginning of the Sky and the Universe they also considered the number."*37 As a result of the applying of the Pythagorean numbers to the reconstruction of being, writes A.F.Losev, a musically-numeral is obtained with spheres, situated on by the other according ti the relations of harmony and numbers."*38

The number öf the Pythagoreans is a way of successfully creating the human reasonable life. "The souls penetrates the body, writes Filolaius, by the number and by the immortal harmony.'*39 The number is the most significant principle of sctructuring of the body and the soul. "Their own numbers, marks A.F. Losev, the Pythagoreans understood as a creative power of being and life, making a way from chaotic potences to the partite, complete and harmonicly wanted organism. That's why the number is being tractated also as a formed, materially organized body , and as a soul, which is considered to be the organizing princip of the human body, and as such mental assumptive, laying in the very basis of the soul itself and in the basis of it's own and of the soul's ideas. "*40

The Pythagoreans discovered symmetry even in the morphological design of leaves, in the position of branches and the leaves on plants, in the design of spineless animals and in every invarietal systems, the existence of which fulfills the cyclic references of the symmetry's axis, spiral under a very strict individual angle for each one.*41 They establish a deep, evristicly significant and in consonance with the modern biological conception, a thouth that "in nature all the geometrically possible symmetry exist."*42

In art, the Pythagorean beliefs about symmetry find their conceptional completeness. The sculptor Poliklet wites his traktat "Kanon", in which he makes the beauty of the human body by the symmetry of its different parts"*43, a a harmonic numeral ration between their magnitudes in strictly fixated tables and graphs, essential for the sculptor's work."The statue of Poliklet "Kanon"is very famous, says Galen: every part of it is proportionate (**συμμετρία**) to the others. The arm spread of a man is equal to the man's height (Mark Virtuvious recreator of Poliklet's numeral proportions). Analyzing Virtuvious's text, A.F. Losev underlines: It's not said that a pentagram is formed by the body, but it's eventually formed. And the pentagram is created based on the laws of the golden cut. "*44 We can only add, that some scientists wrongly display the pentagram as a perfect geometric configuration, which has accepted by the members of the Pythagorean union as their secret logo. B. Rusel pays special attention to the pentagaram's place in the Western European culture. He writes that: "The pentagram has always been associated with magic, and for this responsible are mostly the Pythagoreans, who call it – "health" not pentagram but tetraktidata*45. B and they used it to recognize the secret members of their union- Y.B"*46. G.Yunge more accurately reproduces the place of the pentagram in the Pythagorean culture. "This figure was among great worship with the Pythagoreans and it was a symbol of knowledge /the cursive is mine Y.B./"*47 Two thousand years later Leonardo Da Vinci accepting the Pythagorean understanding for the pentagram displays his renaissance view about the human body in his graphic "The Vitruvian man" – apotheosis of the new human and rational view for the human – it shows the human body by the laws of symmetry and the perfect human body is shown as a pentagram inside a circle's symmetry.

The Pythagoreans put symmetry in the basis of the attitude of the knowing man towards the structure of the world surrounding him. Ïf the numbers was not the essence /of matter/ , Filolai comments, then no one would be able to imagine their essence neither in the attitude towards himself nor in the attitude towards others." But numbers lead everything in the human soul into consonance /in to harmonic, proportional harmonic and symmetric accordance – Y.B./ and by this they make them knowable."*48

For the Pythagoreans symmetry is a criteria for social justice and perfection of the forms of state organization of the greek state cities. Symmetry as harmonic measure, as harmony of measure, overcoming the extreme, sits in the basis of the concept ïsonomia"*49 equality of the ancient citizens before the law. This concept later on the the ancient atomists into a fundamental ontological principle of balance of the equality of the being and not being.*50

In the Pythagorean symmetric model of the world's interpretation enter the medical explanations and health recepies. The Pythagorean Alkemon approves one of the central principles of the ancient medicine, that: "health stores the balance /isonomia/ in the body the forces of wet, dry, cold, warm, bitter, sweet and so on…The Dominancy /monarchia/ of one of them is the reason for an illness. Because the supremacy of one or the other would be fatal.Health is a symmetrical mixture /simmetria krasis/ of such forces."*51 Galen in the treatment of health and he defines it as a certain type of balance with the use of the concept "symmetry" – "the health of the body depends on the proportion (**συμμετρία**) of the warm and cold, dry and wet, from the elements of the body".*52 The same idea is contained in the witneses of Lucian, illustrated not with the concept, but according to isotheral mysteries of the Pythagoreans with perfect holy geometric figure: 'Some of the Pythagoreans and Filolai in that mattr, call the tetrakida /used by them for the greatest oath , because according to their opinion, it forms the secret number 10/ /reason for health/.*53

The Pythagorean views about symmetry as a harmonic measure of the third, middle part, being a function of the two other ending parts of one proportionality, bringing with it the middle ground, overcoming the extremes of this particular proportionality:*54, determines the formation of one significant for the ancient social utopias idea that transaction of the supreme political power to the philosophers is essential for the fulfillment of the perfect, just and guaranteeing prosperity to all governments. The philosopher is a symmetry of the three types of creatures with mind – god, human and creature similar to Pythagoras:*55 This idea will find its final formulation as a symmetric further development of Pythagoras's ideas about man, and human knowledge*56 – Man is the measure for those /matters/ that can be sensed, the mind is the measure to those matters which are senced by the mind."*57

In the naturephylosophical study of Pythagoras the number is presented not only as the most fundamental principle of cosmic organization, the objects which can be sensed, the human soul "which enters the body by the number and harmony:*58 Pythagoras's number is a principle of their symmetrization . This aspect is cached by A.F. Losev:"The numerical harmony creates in the plan of the ancient telling of being: 1/ cosmos with symmetrically placed and tuned into specific musical and numerical tune spheres: 2/ the soul and all things containing harmonic /symmetric Y.B./ structure"*59

At the same time Pythagoras accepts the principle of symmetry as the most fundamental criteria for the numbers' level of perfection. The premium number for him is the decade, for it is presented as a plain figure from of composing spots /equilateral triangle / in his mind symbolizes the perfect symmetry and the essence of the numbers must be view according to the potency which the decade contains."*60 The decade, underlines Filolai, is great and perfect, it fills everything and it's a beginning of the godly, heavenly and human life."*61 Spesvisp- Plato's nephew for his sister Potona's side and his successor as a scholarh in the The Academy next to Ksenocratus"*62 is included in the ancient legend for Plato's purchase of Pythagoras's and Filolaias's works and as author of the tractate: "For the Pythagorean Numbers", based on the basis of carefully studying the Pythagorean lectures and especially Filolai's works:*63, "small book"in almost half of which he interoperates for the decade, explaining it as rooted in nature, best at finishing matters with a number, as a kind of art form /eidos/ with cosmic matters, existing in themselves and belonging to the artist the universal God as the perfect example /paradigma/ "*64. According to the Pythagoreans, for a number to be perfect, correct and in accordance to nature "*65, it has to contain the following: It has to be even, so that the odd and even numbers in it could be at equal count, if the finishing number odd , the odd would be more than the even"*66*, it has to contain equal amount of the first prime and the second non prime numbers "*67, ït has to contain equal amount of factors and productions of factors."*68 and it hs to include all relations: equal, more, lessand so on."*69 Assuming the decade as a perfect number, announces Simplicious, they wished to turn the number of the circular moving bodies to ten. Assuming, says Pythagoras, of sphere of non moving stars, seven spheres of theplanets and earth, they made their numbers to ten with the Antyearth"*70 In this symmetric model, the Pythagoreans assume that the shape of the earth to be a sphere "In the tradition the idea of the earth round shape, says L.Y. Zmudy, is given to Pythagoras. He may have been lead to that thought by symmetry.*71. Being very cautious with the last remark, we can add that the adequate recreation of the idea of symmetry's place the naturephylosophic study for Pythagoras's number makes the scientific conclusion, that he assumes the shape of the Earth, lead to that belief primarily by the ideas of symmetry, because in the logic of his thought the sphere is the most beautiful most wonderful therefore the most symmetric figure.

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